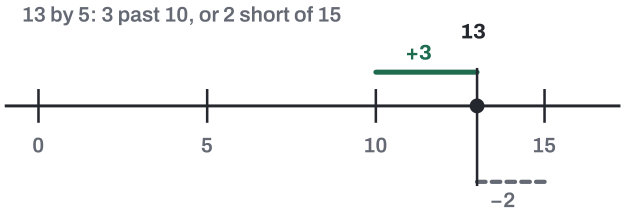
The division form

1 Every number sits between two multiples

Number = divisor × quotient + remainder, and the remainder is always less than the divisor. A number a few past one multiple is also a few short of the next: that is its negative remainder.

Division form	$N = dq + r$, with $0 \le r < d$
Negative remainder	r or $r - d$; a final $-k$ is $d - k$
New divisor kd	$r,\ r + d,\ \ldots,\ r + (k-1)d$
Any new divisor m	Steps of $g = \gcd(d, m)$, starting at r mod g



2 The new divisor is a multiple

$N = 23q + 17$ and $69 = 3 \times 23$: q leaves 0, 1 or 2 by 3.

17 by 23; divided by 69?

$17,\ 17 + 23,\ 17 + 46$

17, 40, 63, all below 69

3 remainders

Trap: The remainder stays 17: 1.

3 Not a multiple: the gcd decides

$20q$ by 32 only hits multiples of $\gcd(20, 32) = 4$.

13 by 20; largest remainder by 32?

13 leaves 1 by 4

Remainders $1, 5, 9, \ldots, 29$

Largest $= 29$

Trap: Divisor minus one: 31.

4 Two divisors, two conditions

Each divisor must beat its own remainder.

5 by d, 41 by $4d$

d divides $41 - 5 = 36$

$d > 5$ and $4d > 41$, so $d \ge 11$

$d = 12, 18, 36$**: 3 values**

Trap: Only $d > 5$: 5 values.

5 Negative remainders

Just below the divisor? Count how far short.

$1997 \times 1998 \times 1999$ by 2000

$(-3)(-2)(-1) = -6$

$-6 + 2000$

Remainder 1994

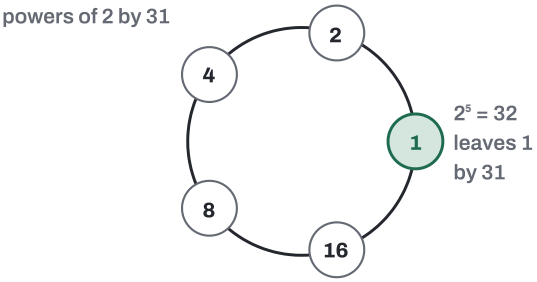
Trap: Dropping the sign: 6.

Powers: the rule of ±1

1 Hunt for a power that leaves 1 or –1

Any power of 1 is 1, and $(-1)^n$ is 1 for even n, -1 for odd n. So find the smallest power of the base that sits one away from a multiple of the divisor, then split the big power around it.

Leaves 1	$a^k = 1$: a^{kq+s} leaves a^s
Leaves -1	$a^k = -1$: a^{kq+s} leaves $(-1)^q a^s$
Shared factor	ax by $am = a \times (x$ by $m)$
Sums	Each term alone, then add



2 Find the power that leaves 1

Split the power around the cycle; keep the spare.

2^{1001} by 31; $2^5 = 32$ leaves 1
$1001 = 5 \times 200 + 1$
$1^{200} \times 2^1$
Remainder 2

Trap: Forgetting the spare 2^1: 1.

3 A power that leaves –1

Count the -1s: odd keeps the minus.

3^{100} by 28; $3^3 = 27$ leaves -1
$100 = 3 \times 33 + 1$
$(-1)^{33} \times 3 = -3$
$-3 + 28 = 25$

Trap: $(-1)^{33}$ read as $+1$: 3.

4 Take out the common factor

Cancel it from both, then multiply it back.

2^{101} by $48 = 16 \times 3$
$2^{101} = 16 \times 2^{97}$
2^{97} by 3: $(-1)^{97} = -1$, so 2
$16 \times 2 = 32$

Trap: Not multiplying back: 2.

5 One term at a time

Each base gets its own sign and parity.

$6^{35} + 8^{36} + 13^{37}$ by 7
$6 = -1, 8 = +1, 13 = -1$
$(-1)^{35} + 1^{36} + (-1)^{37} = -1$
$-1 + 7 = 6$

Trap: Every term as $+1$: 3.

6 Remainders of divisors

Tag each prime of the number by its remainder, $+1$ or -1 (by 3, 2 and 5 are -1, 7 is $+1$). A divisor leaves 1 when its -1 primes have an even total exponent.

Divisors of $700 = 2^2 \times 5^2 \times 7$ leaving 1 by 3
$2^a 5^b$: $a + b$ even, $a, b \in \{0, 1, 2\}$
$(0,0), (0,2), (2,0), (2,2), (1,1)$: 5
7^0 or 7^1: $\times 2$
10 **divisors**

Trap: Half of the 18 divisors: 9.

a + b even lights the cell

	b = 0	b = 1	b = 2
a = 0	1	5	25
a = 1	2	10	50
a = 2	4	20	100

Euler, Fermat, Wilson

1 Euler's number does the hunting for you

$E(N)$ counts the numbers below N that share no factor with N. If a and N share no factor, $a^{E(N)}$ leaves 1, so only the power left after the last multiple of $E(N)$ matters.

Euler's number	$E(N) = N\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots$ over the distinct primes of N
Prime p (Fermat)	$E(p) = p-1$, so a^{p-1} leaves 1
Power on a power	a^{b^c} by p: first find b^c by $E(p)$
Wilson, prime p	$(p-1)!$ leaves $p-1$; $(p-2)!$ leaves 1; $(p-3)!$ leaves $\frac{p-1}{2}$

2 Count the co-primes

Lowest terms: the top shares nothing with 180.

$k/180$ for $k = 1, 2, \ldots, 179$

$180 = 2^2 \times 3^2 \times 5$

$180 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5}$

48 **in lowest terms**

Trap: Overlaps without the multiples of 30: 54.

3 Fermat: a prime's cycle is p – 1

For 19 the cycle is 18, not 19.

5^{75} by 19; 5^{18} leaves 1

$75 = 4 \times 18 + 3$

$5^3 = 125 = 6 \times 19 + 11$

Remainder 11

Trap: Cycle 19, leaving 5^{18}: 1.

4 Euler for a composite divisor

21 is not prime: its cycle is $E(21)$.

2^{100} by $21 = 3 \times 7$

$E(21) = 21 \times \frac{2}{3} \times \frac{6}{7} = 12$

$100 = 8 \times 12 + 4$

$2^4 = 16$

Trap: Cycle 20, as for a prime: 1.

5 A power on a power

Shrink the top first, by the cycle.

3^{5^6} by 7; cycle $E(7) = 6$

5^6 by 6: $5 = -1$, so $(-1)^6 = 1$

3^{5^6} leaves the same as 3^1

Remainder 3

Trap: $(3^5)^6 = 3^{30}$: 1.

6 Wilson for a factorial

Peel the top factors off $30!$.

$28!$ by 31; $30!$ leaves -1

$30! = 28! \times 29 \times 30 = 28! \times (-2)(-1)$

$2 \times 28!$ leaves -1, that is 30

$28!$ **leaves** 15

Trap: The $(p-1)!$ rule: 30.

Finding the number

1 Split, solve, merge

Split a divisor into co-prime parts, find the remainder by each, then list the bigger part's numbers until one fits the other. Walk remainders, not whole lists. Successive division runs the other way: build from the last quotient back.

$an+b$ **divisible by** p	an must leave $-b$: add p to it until a divides
Two remainders	List m_2k+r_2 until it leaves r_1 by m_1; then step by m_1m_2
Successive, least	Last remainder, then $\times$ the divisor before it, $+$ its remainder, back to the first
Successive, all	Least $+$ (product of the divisors) $\times k$

2 Walk the remainder, not the list

Term n is $8n-5$, so $8n$ must leave 5 by 37.

$3, 11, 19, \ldots$; first term divisible by 37

$5, 42, 79, \ldots$: add 37 until 8 divides

$264 = 8 \times 33$

The 33rd term, $259 = 7 \times 37$

Trap: $8n$ leaving -5: the 4th term, 27.

3 Remainder by the product

List the bigger divisor's numbers; stop below 99.

4 by 9, 6 by 11; by 99?

$6, 17, 28, \ldots$ $(11k+6)$

$94 = 10 \times 9 + 4$ fits both

Remainder 94

Trap: Glued as $4 \times 11 + 6$: 50, which leaves 5 by 9.

4 CRT for a power

$91 = 7 \times 13$, two easy cycles.

5^{100} by 7: $5^{100} = 5^{96} \times 5^4$, leaves 2

By 13: $5^2 = -1$, $(-1)^{50} = 1$

$1, 14, 27, \ldots, 79$: 79 leaves 2 by 7

Remainder 79

Trap: Stopping at the piece for 13: 1.

5 Successive: build from the back

Multiply by the divisor before, add that remainder.

By $4, 7, 3$ leaving $1, 5, 2$

$2 \times 7 + 5 = 19$

$19 \times 4 + 1 = 77$

Least number 77

Trap: Each remainder times its own divisor: 78.

6 The step is the product

By 6 then 4 repeats every 24, not 12.

By $6, 4$ leaving $5, 3$: least $3 \times 6 + 5 = 23$

$N = 24k + 23$

$24 \times 40 + 23 \leq 999$

Largest three-digit: 983

Trap: The LCM step 12: 995, which leaves 5 and 1.

7 Swap the order

Write N as step $\times k+$ least; k drops out.

By $4, 5$ leaving $3, 2$: $N = 20k + 11$

By 5: $20k + 11 = 5(4k+2) + 1$

$4k+2$ by 4 leaves 2

Remainders 1 and 2

Trap: The same remainders reversed: 2 and 3.