The numbers tree

1 Know where a number sits

Every CAT number problem starts by placing the number. The named mistakes: 1 is not prime, 0 is even, and "non-negative" includes 0.

> **Real**: rational or irrational
>
> **Rational** $\frac{p}{q}$, $q \neq 0$: integers, fractions, decimals that end or repeat
>
> **Irrational**: decimals that never end or repeat: $\sqrt{2}, \sqrt{3}, \pi$
>
> **Integers**: negative, 0, positive. Natural $1, 2, 3, \ldots$; whole $0, 1, 2, \ldots$
>
> **Primes**: exactly two factors. 2 is the only even one; 25 primes up to 100

Even $\pm$ even	Even; odd $\pm$ odd is even too
Even $\pm$ odd	Odd
Products	Any even factor makes it even; odd $\times$ odd is odd
Powers	Keep the parity: odd^n odd, even^n even
Division	Cannot be decided: $\frac{4}{2} = 2, \frac{6}{2} = 3$

2 The 9s and 0s rule

All the digits minus the fixed part, over one 9 per repeating digit, then one 0 per fixed digit.

$0.3181818\ldots$: 3 fixed, 18 repeats

Top: $318 - 3 = 315$

Bottom: two 9s, one 0: 990

$$\frac{315}{990} = \frac{7}{22}$$

$7 + 22 = 29$

Trap: Not taking the 3 away: $\frac{318}{990} = \frac{53}{165}$, so 218.

3 Which fractions end

In lowest terms, $\frac{1}{n}$ ends exactly when n has no prime but 2 and 5.

n from 2 to 60, $n = 2^a 5^b$

Pure: $2, 4, 8, 16, 32$ and $5, 25$

Mixed: $10, 20, 40, 50$

$7 + 4 = 11$ **values**

Trap: Pure powers only: 7.

4 Three primes need a 2

Three odd primes add to an odd number. An even total forces the 2.

50 as three different primes

$50 = 2 + q + r$, so $q + r = 48$

$5 + 43, 7 + 41, 11 + 37, 17 + 31, 19 + 29$

5 **ways**

Trap: 1 taken as prime, adding $1 + 2 + 47$: 6.

5 Test up to the square root

A composite N has a prime factor no bigger than $\sqrt{N}$. Try only those.

$221, 247, 251, 289, 323$: all under 18^2

Try $2, 3, 5, 7, 11, 13, 17$

$221 = 13 \times 17, 247 = 13 \times 19$

$289 = 17^2, 323 = 17 \times 19$

Only 251: 1 prime

Trap: Stopping at 11: all 5 look prime.

Prime powers, counting factors

1 Every factor is one setting of the power dials

Split into primes first, $N = p^a\, q^b\, r^c$. A factor picks one power of each prime, from 0 up to its top power.

Square	Every power even. Cube: every power a multiple of 3
All factors	$(a+1)(b+1)(c+1)$: the $+1$ is the power 0
Even factors	The 2's power starts at 1: drop the $+1$ on the 2. Odd factors: leave the 2 out
Multiples of k	Each power starts at k's power: $(a - a_k + 1)(b - b_k + 1)\cdots$
Odd count	Only perfect squares: factors pair off, a square's root has no partner

2 Lift every power to a multiple of 3

To make a cube, top each power up to the next multiple of 3.

Boxes of 4500 cubes, stack into one cube

$4500 = 2^2 \times 3^2 \times 5^3$

Need $2^1 \times 3^1$ more

6 **boxes:** $27000 = 30^3$

Trap: Made a square instead: 5.

3 Two rules, one power

Each prime's power must pass both tests at once.

$\frac{n}{2}$ a square, $\frac{n}{3}$ a cube; $n = 2^a 3^b$

$a - 1$ even, a a multiple of 3: $a = 3$

b even, $b - 1$ a multiple of 3: $b = 4$

$n = 8 \times 81 = 648$

Trap: One rule per prime: 72, but $\frac{72}{3} = 24$.

4 Count the dial settings

Add 1 to each power, then multiply.

$10800 = 108 \times 100$

$= 2^4 \times 3^3 \times 5^2$

$(4+1)(3+1)(2+1)$

60 **factors**

Trap: Multiplying the powers: 24.

5 Even factors lock the 2

The 2's power runs from 1, not 0.

$5400 = 2^3 \times 3^3 \times 5^2$

All: $4 \times 4 \times 3 = 48$

Odd: $4 \times 3 = 12$

Even: $3 \times 4 \times 3 = 36$

Trap: Half of all factors: 24.

6 Multiples of k: fix the forced powers

Start each power at k's power, then count up to N's.

$21600 = 2^5 \times 3^3 \times 5^2$, $12 = 2^2 \times 3$

2: powers 2 to 5, 4 ways

3: 1 to 3, 3 ways; 5: 0 to 2, 3 ways

$4 \times 3 \times 3 = 36$

Trap: Subtracting with no $+1$: $3 \times 2 \times 3 = 18$.

7 Odd count means a square

Factors come in pairs d and $\frac{N}{d}$. Only a square has a middle one.

Numbers from 1 to 200

Odd count $\Leftrightarrow$ perfect square

$1^2, 2^2, \ldots, 14^2 = 196$

14 **numbers**

Trap: Forgetting 1 is a square: 13.

Factor shapes, sums, products

1 Read a factor count backwards

Split the count every way it splits; each piece is a power plus one. For the smallest number, put the biggest powers on the smallest primes.

Exactly 2, 3, 4	p; p^2; p^3 or $p\,q$
Exactly 6	p^5 or p^2q
Sum of factors	$(1+p+\cdots+p^a)(1+q+\cdots+q^b)\cdots$, each bracket $\frac{p^{a+1}-1}{p-1}$
Product of factors	$N^{d/2}$, d the number of factors

2 Cube factors: stops at 0, 3, 6

Split composite bases first, then let each power stop only on multiples of 3.

$12^3 \times 15^4 = 2^6 \times 3^3 \times 3^4 \times 5^4$

$= 2^6 \times 3^7 \times 5^4$

2: $0, 3, 6$; 3: $0, 3, 6$; 5: $0, 3$

$3 \times 3 \times 2 = 18$

Trap: 12 and 15 treated as primes: 4.

3 Exactly 6 factors: two shapes

$6 = 6$ or 3×2: count both.

Numbers below 60

p^5: 32

p^2q: $12, 20, 28, 44, 52, 18, 45, 50$

$1 + 8 = 9$

Trap: Missing p^5: 8.

4 Smallest with 18 factors

Try every split, biggest power on 2.

9×2: $2^8 \times 3 = 768$

6×3: $2^5 \times 3^2 = 288$

$3 \times 3 \times 2$: $2^2 \times 3^2 \times 5 = 180$

Smallest: 180

Trap: Stopping at two primes: 288.

5 Sum of even factors

Write each prime's powers as a sum; leave $2^0 = 1$ out.

$360 = 2^3 \times 3^2 \times 5$

$(2+4+8)(1+3+9)(1+5)$

$= 14 \times 13 \times 6$

1092

Trap: Keeping 2^0, the sum of all: 1170.

6 Multiples of 6: pull the 6 out

Factors of N that are multiples of 6 are $6\times$ the factors of $\frac{N}{6}$.

720, so $\frac{720}{6} = 120 = 2^3 \times 3 \times 5$

Sum for 120: $15 \times 4 \times 6 = 360$

$6 \times 360 = 2160$

Trap: Not multiplying the 6 back: 360.

7 Product of factors pairs off

Pairs multiply to N; a square writes as a power of its root.

$400 = 2^4 \times 5^2$: $5 \times 3 = 15$ factors

Product $= 400^{15/2}$

$= (20^2)^{15/2}$

20^{15}, **so** $k = 15$

Trap: 400^{15}, no halving: 30.

Factor pairs

1 Name the variables, then count

Factors come in pairs. Named variables (x and y, tap A and tap B) mean order counts. d: number of factors; k: number of different primes.

Pairs $a \times b = N$	Ordered: d. Unordered: $\frac{d}{2}$; for a square $\frac{d+1}{2}$, or $\frac{d-1}{2}$ if the two must differ
Co-prime pairs	2^{k-1} unordered, 2^k ordered: each prime's full power goes to one side
$x^2 - y^2 = N$	$(x-y)(x+y) = N$: both factors the same parity
$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$	$(x-n)(y-n) = n^2$: one ordered pair per factor of n^2

2 Rectangles from tiles

A square N has an odd number of factors; its middle pair is the square.

$3600 = 2^4 \times 3^2 \times 5^2$: 45 factors

Unordered pairs: $\frac{45+1}{2} = 23$

Not a square: drop 60×60

22 rectangles

Trap: Counting the square: 23.

3 Co-prime pairs: whole powers move

A shared prime breaks co-primality, so each power goes wholly to one side.

$1800 = 2^3 \times 3^2 \times 5^2$: blocks $8, 9, 25$

Each block left or right: $2^3 = 8$ ordered

$1 \times 1800, 8 \times 225, 9 \times 200, 25 \times 72$

4 ways

Trap: Counting order: 8.

4 Difference of squares: same parity

$x - y$ and $x + y$ differ by $2y$, an even number.

$x^2 - y^2 = 96$: six factor pairs of 96

Both even: $2 \cdot 48, 4 \cdot 24, 6 \cdot 16, 8 \cdot 12$

$(25, 23), (14, 10), (11, 5), (10, 2)$

4 pairs

Trap: Ignoring parity: 6.

5 Two taps: shift by 12

$\frac{1}{a} + \frac{1}{b} = \frac{1}{12}$ clears to $ab = 12a + 12b$, then add 144 to both sides.

$(a-12)(b-12) = 144$

$144 = 2^4 \times 3^2$: $5 \times 3 = 15$ factors

Tap A and tap B are named: ordered

15 pairs, from $(13, 156)$ to $(156, 13)$

Trap: A and B not told apart: 8.

6 Divide out the remainder

Split the top as (divisor) $\times$ something $+$ a number; the divisor must divide that number.

$n^2 + 5n + 72 = (n+2)(n+3) + 66$

$n + 2$ divides 66: $1, 2, 3, 6, 11, 22, 33, 66$

$n \geq 1$, so $n + 2 \geq 3$

6 values

Trap: Keeping 1 and 2: 8.