Last digits and digit sums

1 Know which digits to look at

Each test also gives the **remainder**. Missing digits run 0 to 9: list the multiples in reach.

$2, 4, 8, 16$	2^k: the last k digits
$5, 25, 125$	5^k: the last k digits
125	Ends $000, 125, \ldots, 875$
10^k	k zeros at the end
3 **and** 9	Digit sum, in any order

Remainder of $4{,}817{,}203{,}916$ by $16 = 2^4$

Last four digits: 3916

$3916 = 16 \times 244 + 12$

Remainder 12

Trap: Three digits, as for 8: 916 leaves 4.

2 8 looks at three digits

Only $K72$ matters, and $100K$ leaves $4K$.

$531K72$ divisible by 8

$100K + 72 = 96K + 4K + 72$

$4K + 72$ a multiple of 8: K even

$K = 0, 2, 4, 6, 8$: 5 **values**

Trap: The 4 test on 72 passes every K: 10.

3 125 has eight endings

$375 = 125 \times 3$: three digits first, then the digit sum.

$9A3B5$ divisible by 375

$3B5$ must be 375: $B = 7$

$9 + A + 3 + 7 + 5 = 24 + A$

$A = 0, 3, 6, 9$: 4 **numbers**

Trap: 25's two-digit test lets $B = 2$ in: 7.

4 Two missing digits for 9

$A + B \leq 18$: try every multiple of 9 in reach.

$52A61B$ divisible by 9

$5 + 2 + 6 + 1 = 14$

$14 + A + B = 18$ or 27

$A + B = 4$: 5 pairs; $A + B = 13$: 6 pairs

11 **pairs**

Trap: Only $A + B = 4$: 5.

5 Order doesn't change a digit sum

All orders of the same digits pass or fail together.

Seven-digit numbers using 1 to 7 once each

$1 + 2 + \cdots + 7 = 28$

28 leaves 1 on division by 3

0 **are divisible by** 3

Trap: One in three of $7! = 5040$: 1680.

6 Casting out nines finds a hidden digit

A number leaves the same remainder by 9 as its digit sum. Digit sums **multiply** when the numbers multiply.

$123456 \times 79 = 975A024$

123456: 21, leaves 3; 79: 16, leaves 7

Product leaves $3 \times 7 = 21$, so 3

$9 + 7 + 5 + A + 0 + 2 + 4 = 27 + A$

$A = 3$

Sum	Digit sums add
Product	Digit sums multiply
Check	$123456 \times 79 = 9{,}753{,}024$

Trap: Adding $3 + 7 = 10$, so 1: $A = 1$.

11, 99, 999, 1001

1 Blocks from the right

k-digit blocks **add** for $10^k - 1$ and **alternate** for $10^k + 1$. Always group from the right.

$9, 99, 999$	Add digits, pairs, triplets; $999 = 27 \times 37$ serves 27 and 37
$11, 101, 1001$	Alternate digits, pairs, triplets; $1001 = 7 \times 11 \times 13$ serves all three

31415926535 by 99

3	14	15	92	65	35	from here

3 + 14 + 15 + 92 + 65 + 35 = 224

224: 2 + 24 = 26, the remainder

Remainder of $31{,}415{,}926{,}535$ by 99

$35 + 65 + 92 + 15 + 14 + 3 = 224$

Again: $2 + 24 = 26$

Remainder 26

Trap: Pairs from the left $(31, 41, 59, \ldots)$: 17.

2 11 with two missing digits

The alternating sum can be 0 or ± 11.

$7A5B3$ divisible by 11

$3 - B + 5 - A + 7 = 15 - (A + B)$

$A + B = 15$: 4 pairs

$A + B = 4$: 5 pairs

9 **pairs**

Trap: Only the "$= 0$" case: 4.

3 A number plus its reverse

$\overline{ab} + \overline{ba} = 11(a + b)$, $\overline{ab} - \overline{ba} = 9(a - b)$.

$\overline{ab} + \overline{ba}$ a perfect square

$11(a + b)$ square, $a + b \leq 18$

$a + b = 11$: $29, 38, \ldots, 92$

8 **numbers**

Trap: 29 and 92 as one: 4.

4 999 means 27 and 37

Count the digits: a leftover block still counts.

Ten 4s by 37: $4 \mid 444 \mid 444 \mid 444$

$4 + 3 \times 444 = 1336$

$1 + 336 = 337 = 37 \times 9 + 4$

Remainder 4

Trap: $444 = 12 \times 37$, the lone 4 forgotten: 0.

5 1001 means 7, 11 and 13

Alternate the triplets, then divide what is left.

$5{,}830{,}617{,}294$ by 13

$294 - 617 + 830 - 5 = 502$

$502 = 13 \times 38 + 8$

Remainder 8

Trap: Adding the triplets (the 999 rule): 6.

6 abcabc is abc × 1001

7, 11 and 13 come free.

$\overline{abcabc}$ divisible by 364

$\overline{abcabc} = \overline{abc} \times 7 \times 11 \times 13$

$364 = 4 \times 7 \times 13$

$\overline{abc}$ a multiple of 4: $100, 104, \ldots, 996$

225 **numbers**

Trap: $\overline{abc}$ itself a multiple of 364: 2.

7 101 alternates pairs

Pairs from the right, signs $+, -, +, -$.

$20{,}251{,}231$ by 101

Pairs: $20 \mid 25 \mid 12 \mid 31$

$31 - 12 + 25 - 20 = 24$

Remainder 24

Trap: Adding the pairs (the 99 rule): 88.

Composite divisors

1 Split into co-prime parts

Test each part. The parts must share no factor: $24 = 8 \times 3$, never 4×6. Do the part with fewer cases first.

$12,\ 18,\ 24$	$3 \times 4,\ 2 \times 9,\ 3 \times 8$
$36,\ 72$	$4 \times 9,\ 8 \times 9$
$88,\ 99$	$8 \times 11,\ 9 \times 11$
Both a and b	The LCM, not $a \times b$

$315K24$ divisible by $24 = 8 \times 3$

8: $K24$, so K even

3: $15 + K$, so $K = 0, 3, 6, 9$

$K = 0$ or 6: 2 values

Trap: 4×6 lets $K = 3, 9$ in (315324 is $12\times$ odd): 4.

2 88 is 8 × 11

8 pins the last-digits unknown, 11 ties the two.

$3A61B8$ divisible by 88

8: $1B8 = 128$ or 168, so $B = 2, 6$

11: $8 - B + 1 - 6 + A - 3 = A - B$

$A = B$: $326128, 366168$

Largest 366168

Trap: The 4 test on $B8$ lets $B = 8$ in: 386188.

3 6 from a set of digits

Digit sets for 3 first, then an even digit last.

Different digits from 1 to 5, divisible by 6

Sum a multiple of 3:
$\{1, 2, 3\}, \{1, 3, 5\}, \{2, 3, 4\}, \{3, 4, 5\}$

Even digit last: $2 + 0 + 4 + 2$

8 numbers

Trap: The 3 test only, $4 \times 3!$: 24.

4 Both means the LCM

A shared factor is counted once.

Multiples of 6 and of 15 in 100 to 999

$\text{LCM}(6, 15) = 30$

$\lfloor 999/30 \rfloor - \lfloor 99/30 \rfloor = 33 - 3$

30 numbers

Trap: Multiples of $6 \times 15 = 90$: 10.

5 Sharing no factor: keep the survivors

When N is a multiple of every prime, strike them out one by one.

1 to 210, no common factor with $42 = 2 \times 3 \times 7$

Not even: $210 \times \frac{1}{2} = 105$

Not by 3: $105 \times \frac{2}{3} = 70$

Not by 7: $70 \times \frac{6}{7} = 60$

60 numbers

Trap: Inclusion–exclusion without the triple term: 65.

Expressions

1 Powers of a and b

Read $a \equiv -b \pmod{a+b}$ and $a \equiv b \pmod{a-b}$: the rules follow from the sign of $(-1)^n$.

$a^n - b^n$	By $a-b$ always; by $a+b$ when n is even
$a^n + b^n$	By $a+b$ when n is odd
Fermat	$n^p - n$ is a multiple of a prime p
In a row	k consecutive integers: a multiple of $k!$

$11^n + 5^n$ by 16, $n = 1$ to 20

$11 \equiv -5 \pmod{16}$

Odd n: $-5^n + 5^n = 0$

Even n: 2×5^n, never a multiple of 16

10 **values**

Trap: $a+b$ assumed to divide for every n: 20.

2 a – b always, a + b only for even n

Odd power with a minus sign: $a-b$ only.

Which of $32, 28, 14, 9$ divides $23^{15} - 9^{15}$?

$a - b = 14$ always

$a + b = 32$ needs n even; 15 is odd

$23 \equiv 3, 9 \equiv 1 \pmod{4}$: not 28

14

Trap: $a+b$ with an odd power: 32.

3 The number that divides every value

Plug in small values, take the gcd, then factorise to see why.

Largest number dividing every $n^5 - n$

$n = 2$: 30; $n = 3$: 240

$\gcd(30, 240) = 30$

$(n-1)n(n+1)(n^2+1)$: 2, 3, and 5 by Fermat

30

Trap: Only the three in a row: 6.

4 Odd n in the middle

Two even neighbours: one is a multiple of 4.

$(n-1)n(n+1)$ for every odd $n \geq 3$

$n-1, n+1$ even, one a multiple of 4: 8

Three in a row: 3

$n = 3$ gives 24 itself

Largest divisor 24

Trap: Three in a row, 3!: 6.

5 The power of 2 in a sum

Take the smallest power out, then look inside.

$4^9 + 8^6 + 16^5$

$= 2^{18} + 2^{18} + 2^{20}$

$= 2^{18}(1 + 1 + 4) = 2^{18} \times 6$

$= 2^{19} \times 3$

$x = 19$

Trap: The smallest power alone: 18.