Unit digits

1 Only the last digits play, in cycles of 4

A sum or product's unit digit comes from the unit digits alone. Powers repeat every 4: divide the power by 4 and keep the remainder. Remainder 0 means the **4th** power.

Fixed	$0, 1, 5, 6$ never change
Power $= 4k$	Even last digit: 6. Odd, not 5: 1
Power by 4	Read only its last two digits
Subtracting	A smaller digit on top borrows 10

last digit	power 1	2	3	4
2	2	4	8	**6**
3	3	9	7	**1**
4	4	6	4	**6**
7	7	9	3	**1**
8	8	4	2	**6**
9	9	1	9	**1**

2 Remainder 0 means the 4th power

The cycle has slots 1 to 4, no slot 0.

Units digit of 4738^{2028}
Last digit 8; $2028 = 4 \times 507$
Remainder 0: use $8^4 = 4096$
Units digit 6

Trap: $8^0 = 1$: 1.

3 The power's last two digits

Divisibility by 4 needs two digits, not one.

Units digit of 1987^{3458}
Last two digits of the power: $58 = 4 \times 14 + 2$
Use $7^2 = 49$
Units digit 9

Trap: "Ends in 8, a multiple of 4", so 7^4: 1.

4 A difference borrows

Never write a negative unit digit.

Units digit of $8^{43} - 3^{38}$
43 leaves 3: $8^3 = 512$, ends in 2
38 leaves 2: $3^2 = 9$
$2 - 9$ borrows: $12 - 9$
Units digit 3

Trap: $9 - 2$: 7.

5 A power tower

Only the tower's remainder by 4 is needed.

Units digit of $17^{19^{21}}$
$19 = 20 - 1$, so 19^{21} is 1 short of a multiple of 4
Remainder 3: $7^3 = 343$
Units digit 3

Trap: "Odd power, remainder 1", so 7^1: 7.

6 Which n make a sum end in 0

Only n mod 4 matters, so check four cases and count.

$1^n + 2^n + 3^n + 4^n$, $n = 1$ to 100
Remainder 1: $1 + 2 + 3 + 4 = 10$
Remainder 2: $1 + 4 + 9 + 6 = 20$
Remainder 3: $1 + 8 + 7 + 4 = 20$
Remainder 0: $1 + 6 + 1 + 6 = 14$, fails
$100 - 25 = 75$ **sums end in** 0

Three of every four n work. The multiples of 4 are the only ones that fail.

Trap: Counting the multiples of 4: 25.

Zeros at the end

1 A zero is one 2 and one 5

Zeros at the end $= \min(2\text{s}, 5\text{s})$. In $n!$ the 5s always run out first, so count them: divide by 5, then the quotient by 5, and add.

Zeros of $n!$	$\lfloor n/5 \rfloor + \lfloor n/25 \rfloor + \lfloor n/125 \rfloor + \cdots$
Jumps	One more at each multiple of 5, two at 25, three at 125: some counts never happen
At least k zeros	Start at $4k$, count, step up by fives
Not a factorial	Count both 2s and 5s: the 2s can run out

Zeros at the end of $260!$

$260 \div 5 = 52$

$52 \div 5 = 10$

$10 \div 5 = 2$

$52 + 10 + 2 = 64$ **zeros**

Trap: Counting the multiples of 10: 26.

2 Smallest n with at least k zeros

Multiply by 4, not 5, then test.

At least 50 zeros

$50 \times 4 = 200$: $40 + 8 + 1 = 49$

One more 5 needed: 205

205: $41 + 8 + 1 = 50$

$n = 205$

Trap: 50×5: 250.

3 The count that is skipped

125 brings three 5s at once.

Exactly 30 zeros?

$124!$: $24 + 4 = 28$

$125!$: $25 + 5 + 1 = 31$

29 and 30 are skipped

0 **values of** n

Trap: "Always a block of five": 5.

4 A product of evens

Pull a 2 out of every factor first.

$2 \times 4 \times 6 \times \cdots \times 200$

$= 2^{100} \times (1 \times 2 \times \cdots \times 100)$

$= 2^{100} \times 100!$

5s of $100!$: $20 + 4$

24 **zeros**

Trap: The 5s of $200!$: 49.

5 When the 2s run out

Outside a factorial, count both.

$20^{15} \times 25^{12}$

$20^{15} = 2^{30} \times 5^{15}$

$25^{12} = 5^{24}$

$2^{30} \times 5^{39}$: $\min(30, 39)$

30 **zeros**

Trap: Counting the 5s by habit: 39.

The highest power in n!

1 Divide again and again, add the quotients

For a **prime** p, the power of p in $n!$ is $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \cdots$. For a **composite**, split it into prime powers, count each prime, divide by its power, and take the smallest.

Prime p	Successive division of n by p
p^2 **inside**	The count of p, halved (round down)
Composite	Smallest of the counts; not always the largest prime
A run	$(m+1) \times \cdots \times n = \frac{n!}{m!}$: subtract the counts

Largest k with 7^k dividing $500!$

$500 \div 7 = 71$

$71 \div 7 = 10$

$10 \div 7 = 1$

$k = 71 + 10 + 1 = 82$

Trap: One division only: 71.

2 A squared prime goes in pairs

$18 = 2 \times 3^2$: the 3s are halved.

Largest k with 18^k dividing $40!$

2s: $20 + 10 + 5 + 2 + 1 = 38$

3s: $13 + 4 + 1 = 18$, in pairs 9

$\min(38, 9)$

$k = 9$

Trap: The count of 3s itself: 18.

3 The bigger prime may not decide

$24 = 2^3 \times 3$: three 2s per 24.

Largest k with 24^k dividing $30!$

2s: $15 + 7 + 3 + 1 = 26$, in threes 8

3s: $10 + 3 + 1 = 14$

$\min(8, 14)$

$k = 8$

Trap: "The largest prime decides": 14.

4 A run is n! over m!

Multiples of 25 inside bring extra 5s.

Largest k with 5^k dividing $101 \times \cdots \times 250$

$= \frac{250!}{100!}$

5s: $250!$ has $50 + 10 + 2 = 62$

$100!$ has $20 + 4 = 24$

$k = 62 - 24 = 38$

Trap: Just the multiples of 5 in the run: 30.

5 Smallest n for a prime power

$p \times k$ overshoots: multiples of p^2 bring extra factors.

Smallest n with 3^{20} dividing $n!$

60: $20 + 6 + 2 = 28$, too many

45: $15 + 5 + 1 = 21$

44: $14 + 4 + 1 = 19$

$n = 45$

Trap: 3×20: 60.

HCF and LCM

1 HCF divides, LCM is divided

HCF $\leq$ the numbers $\leq$ LCM. HCF takes each shared prime at its lowest power, LCM each prime at its highest.

Two numbers	HCF $\times$ LCM $=$ product
HCF, remainders	Take each remainder off, then HCF. Unknown but equal: HCF of the differences
LCM, remainders	Same r: LCM $\cdot k + r$. Each d short: LCM $- d$
Together again	Every LCM of the intervals or steps

Largest divisor of $167, 342, 587$ leaving one remainder

Differences: $175, 245$

HCF$(175, 245) = 35$

35 **(each leaves** 27**)**

Trap: One difference only: 175.

2 Different remainders

Take each one off its own number.

$398, 436, 542$ leave $7, 11, 15$

$391, 425, 527$

$17 \times 23, 17 \times 25, 17 \times 31$

17 **per box**

Trap: HCF of the differences: 2.

3 LCM + r, then one more condition

Walk the list until it divides.

Trays of $6, 8, 15$ leave 2; trays of 11 leave none

LCM $= 120$: $122, 242, \ldots$

$242 = 11 \times 22$

242 **eggs**

Trap: LCM $+ 2$ unchecked: 122.

4 All short by the same amount

Each remainder is 2 short of its divisor.

Divisors $9, 12, 15$; remainders $7, 10, 13$

LCM$(9, 12, 15) = 180$

$180 - 2$

The number is 178

Trap: LCM $+ 2$: 182.

5 Together again

The starting moment counts.

Lights every $48, 72, 108$ s, for one hour

LCM $= 2^4 \times 3^3 = 432$ s

$3600 \div 432$: 8 steps

$8 + 1 = 9$ **times**

Trap: Forgetting the start: 8.

6 Pairs from HCF and LCM

Numbers $6a, 6b$ with a, b co-prime.

HCF 6, LCM 360

$a \times b = 360 \div 6 = 60 = 2^2 \times 3 \times 5$

Each block $4, 3, 5$ goes whole to one side

$2^{3-1} = 4$ **pairs**

Trap: Every factor pair of 60: 6.

7 Common terms step by the LCM

Not by the product of the steps.

$5, 11, 17, \ldots, 401$ and $3, 11, 19, \ldots, 403$

First common: 11; step LCM$(6, 8) = 24$

$11 + 24k \leq 401$: $k = 0$ to 16

17 **common terms**

Trap: Stepping by 48: 9.