Lines and angles

1 Parallels: draw one more

A transversal across two parallels makes only two sizes of angle, x and $180^\circ - x$: alternate (Z) and corresponding (F) angles are equal, co-interior (C) angles add to 180°. At a bend between parallels, draw a third parallel through the bend.

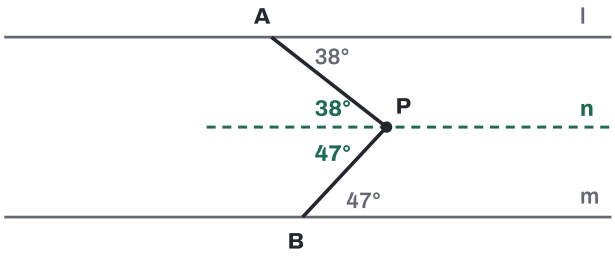
$l \parallel m$: 38° at A, 47° at B

Draw $n \parallel l$ through P

Upper part of $\angle P$: 38° (alternate)

Lower part of $\angle P$: 47° (alternate)

$\angle APB = 38^\circ + 47^\circ = 85^\circ$



Trap: Closing it as a triangle: $180^\circ - 85^\circ = 95^\circ$.

2 Many bends: right = left

A parallel through every bend: angles opening right add up to angles opening left, counting the angles at l and m.

A: 30° opens right; P: 75° opens left

Q: x opens right; B: 50° opens left

$30^\circ + x = 75^\circ + 50^\circ$

$x = 95^\circ$

Trap: Dropping the 30° at A: 125°.

3 Exterior angle, and the star

An exterior angle equals the two interior angles opposite it. In a five-pointed star, two exterior angles carry four tips into one triangle: the five tips add to 180°.

Four tips: $30 + 35 + 40 + 45 = 150$

Fifth tip $= 180^\circ - 150^\circ = 30^\circ$

Trap: 360° belongs to the six-pointed star (two whole triangles): 210°.

4 Isosceles: apex or base?

The given angle can be the apex or a base angle. Try both.

One angle is 40°

Apex 40°: $40^\circ, 70^\circ, 70^\circ$

Base 40°: $40^\circ, 40^\circ, 100^\circ$

Largest: $70^\circ + 100^\circ = 170^\circ$

Trap: One case only: 100°.

5 Equal sides, then the exterior angle

Equal sides face equal angles; each exterior angle doubles the one before.

$AB = AC$, $AD = BD = BC$, $\angle A = x$

$AD = BD$, so exterior $\angle BDC = x + x = 2x$

$BD = BC$: $\angle C = 2x$, so $\angle B = 2x$

$x + 2x + 2x = 180^\circ$

$x = 36^\circ$

Trap: $\angle BDC$ taken as x: 60°.

6 Equidistant from three corners

If the midpoint M of BC has $MA = MB = MC$, then $\angle A = 90^\circ$: the median to the hypotenuse is half of it.

$MA = MB = MC$, so $\angle A = 90^\circ$

$\angle B + \angle C = 90^\circ$, $\angle B = 3\angle C$

$4\angle C = 90^\circ$, $\angle C = 22.5^\circ$

$\angle B = 67.5^\circ$

Trap: $\angle B + \angle C = 180^\circ$: 135°.

Sides and right triangles

1 What three sides allow

Any triangle	$b - a < c < a + b$, never equal: a flat triangle is not a triangle
Longest side c	$c^2 < a^2 + b^2$ acute, $c^2 = a^2 + b^2$ right, $c^2 > a^2 + b^2$ obtuse
Bigger side	Faces the bigger angle; equal sides face equal angles
Triplets	$(3, 4, 5)$, $(5, 12, 13)$, $(8, 15, 17)$, $(7, 24, 25)$, $(20, 21, 29)$ and their multiples
Set squares	$45°$–$45°$–$90°$: $1 : 1 : \sqrt{2}$ · $30°$–$60°$–$90°$: $1 : \sqrt{3} : 2$

2 Flat triangles don't count

Longest side n, middle side b: the shortest runs from $n - b + 1$ to b.

$n = 9$: $b = 5$ to 9 give $1, 3, 5, 7, 9$

$1 + 3 + 5 + 7 + 9 = 25$ **triangles**

Odd n: $\left(\frac{n+1}{2}\right)^2$

Trap: Allowing $a + b = 9$: 29.

3 Obtuse: two ways

The obtuse angle can face the new side or the old longest one.

Sides 8, 15, c: $7 < c < 23$

$c^2 > 8^2 + 15^2 = 289$: $c = 18$ to $22, 5$

$c^2 < 15^2 - 8^2 = 161$: $c = 8$ to $12, 5$

$5 + 5 = 10$ **values**

Trap: Only the first case: 5.

4 Median and altitude to the hypotenuse

The median to the hypotenuse is half of it (its midpoint is the centre of the circle through the corners). The altitude is $\frac{ab}{c}$.

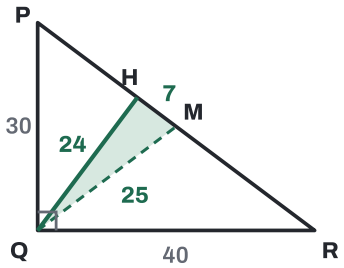
$\angle Q = 90°$, $PQ = 30$, $QR = 40$

$PR = \sqrt{30^2 + 40^2} = 50$, median $QM = \frac{50}{2} = 25$

Altitude $QH = \frac{30 \times 40}{50} = 24$

$HM = \sqrt{25^2 - 24^2}$

$HM = 7$ **cm**



Trap: Subtracting the lengths: $25 - 24 = 1$.

5 Split at a $45°$ and a $30°$ corner

Drop the altitude: the $45°$ half is $1 : 1$, the $30°$ half has base $= \sqrt{3} \times$ height.

$\angle B = 45°$, $\angle C = 30°$, altitude $AD = 6$

$45°$ half: $BD = 6$

$30°$ half: $DC = 6\sqrt{3}$

$BC = 6 + 6\sqrt{3}$

Area $= \frac{1}{2} \times (6 + 6\sqrt{3}) \times 6$

$18 + 18\sqrt{3}$ **sq cm**

Trap: $DC = \frac{6}{\sqrt{3}}$, the ratio upside down: $18 + 6\sqrt{3}$.

Area

1 One area, many ways to get it

Any side as base	$\frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$, so the longest side has the shortest altitude
Three sides	Heron: $\sqrt{s(s-a)(s-b)(s-c)}$ with $s = \frac{a+b+c}{2}$
Equilateral	Area $\frac{\sqrt{3}}{4}a^2$, height $\frac{\sqrt{3}}{2}a$
Same height	Areas are in the ratio of the bases
Shared angle	Area ratio = product of the two side ratios at that angle

2 Same height, area follows the base

D on BC with $BD : DC = 3 : 1$, E on AD with $AE : ED = 2 : 1$, area $ABC = 96$.

$$ADC = \frac{1}{4} \times 96 = 24$$

$$EDC = \frac{1}{3} \times 24$$

8 **sq cm**

Trap: Using AE's two parts: 16.

3 Cut off three corners

D, E, F a third of the way along BC, CA, AB, going round; area 63.

Each corner: $\frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$

Three corners: $\frac{6}{9}$, so $DEF = \frac{3}{9} \times 63$

21 **sq cm**

Trap: DEF as a $\frac{1}{3}$ copy: $\frac{1}{9} \times 63 = 7$.

4 Medians make six equal pieces

Three medians cut a triangle into six equal areas; the centroid G cuts each median $2 : 1$ from the corner.

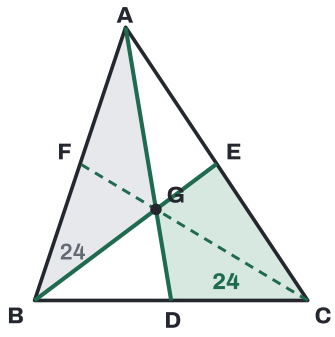
Medians AD, BE meet at G; $ABG = 24$

ABG is two pieces: one piece $= 12$

$GDCE$ is two pieces

$GDCE = 24$ **sq cm**

Trap: $GD = \frac{1}{2}AG$, so half the area: 12.



5 Heron, then the shortest altitude

Sides $10, 17, 21$: $s = 24$

Area $= \sqrt{24 \times 14 \times 7 \times 3} = 84$

Longest side, shortest altitude

$$h = \frac{2 \times 84}{21}$$

8 **cm**

Trap: On the shortest side: 16.8.

6 Inside an equilateral triangle

Join the point to the corners: three triangles on equal bases, so the distances add to the height $\frac{\sqrt{3}}{2}a$.

Side 12; distances 2 and 3

$$\text{Sum} = \frac{\sqrt{3}}{2} \times 12 = 6\sqrt{3}$$

Third $= 6\sqrt{3} - 5 \approx 5.39$ **cm**

Trap: Adding to the side: $12 - 5 = 7$.

The four centres

1 Which lines meet where

Centroid G	Medians; $AG : GD = 2 : 1$; six equal areas
Incentre I	Angle bisectors; $r = \frac{\text{area}}{s}$; $\angle BIC = 90^\circ + \frac{A}{2}$
Circumcentre O	Perpendicular bisectors of the sides; $R = \frac{abc}{4 \times \text{area}}$
Orthocentre H	Altitudes; $\angle BHC = 180^\circ - A$ in an acute triangle ($A = 64^\circ$: 116°, not the incentre's 122°)
Right triangle	$r = s - h$ and $R = \frac{h}{2}$, with h the hypotenuse
Equilateral	All four are one point; $r = \frac{a}{2\sqrt{3}}$, $R = \frac{a}{\sqrt{3}} = 2r$

2 Bisector: the ratio of the arms

$AB = 12$, $AC = 18$, $BC = 20$

$BD : DC = AB : AC = 2 : 3$

$BD = \frac{2}{5} \times 20 = 8$ **cm**

Trap: The ratio reversed: 12.

3 Where I cuts the bisector

$AB = 13$, $AC = 14$, $BC = 15$

$BD = \frac{13}{27} \times 15 = \frac{65}{9}$

BI bisects $\angle B$: $AI : ID = 13 : \frac{65}{9}$

$AI : ID = (13 + 14) : 15 = 9 : 5$

Trap: The centroid's $2 : 1$.

4 Two bisectors meet at $90^\circ + \frac{A}{2}$

$\angle QIR = 180^\circ - \frac{Q + R}{2} = 90^\circ + \frac{P}{2}$

$90^\circ + \frac{P}{2} = 3P$

$P = 36^\circ$

Trap: $180^\circ - P = 3P$: 45°.

5 Right triangle: $r = s - h$

Perimeter 70, so $s = 35$; $r = 6$

$h = 35 - 6 = 29$

Check $20, 21, 29$: $\frac{1}{2} \times 20 \times 21 \div 35 = 6$

Hypotenuse 29 **m**

Trap: $70 - 6 = 64$.

6 $R = \frac{abc}{4 \times \text{area}}$

$10, 10, 12$: height $\sqrt{10^2 - 6^2} = 8$, area 48

$R = \frac{10 \times 10 \times 12}{4 \times 48}$

$R = \frac{25}{4} = 6.25$ **cm**

Trap: Half of BC, as if right-angled: 6.

7 Equilateral: one centre, two circles

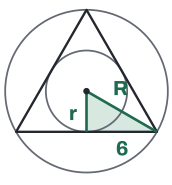
R, r and half a side make a right triangle.

$R^2 - r^2 = \left(\frac{a}{2}\right)^2$

Ring $= \pi(R^2 - r^2)$

$= \pi \times 6^2$

36π **sq cm**



Trap: $\pi(R - r)^2 = 12\pi$.