Parallelograms

1 One family, one extra rule each

Every shape keeps all of its parent's rules and adds one.

Parallelogram	Opposite sides equal and parallel · opposite angles equal · neighbours add to 180° · diagonals bisect each other
Rectangle	+ every angle 90° · diagonals equal: $d^2 = a^2 + b^2$
Rhombus	+ all sides equal · diagonals cross at 90° · area $\frac{1}{2}d_1 d_2$ · $d_1^2 + d_2^2 = 4a^2$
Square	All of it · diagonal $= a\sqrt{2}$ · area $= a^2 = \frac{d^2}{2}$
Trapezium	One pair of parallel sides · area $= \frac{1}{2}(a+b)\,h$
Isosceles trapezium	+ legs equal, base angles equal · area $= BX \times h$ (page 4)

2 Neighbours add to 180°

Opposite angles are equal; neighbouring angles are co-interior.

$\angle B$ is 24° more than half of $\angle A$

$$A + \left(\frac{A}{2} + 24\right) = 180$$

$$\frac{3}{2}A = 156$$

$$\angle A = 104^\circ$$

Trap: Neighbours taken as equal: $A = \frac{A}{2} + 24$ gives 48°.

3 The parallelogram law

The squares of the two diagonals add to twice the squares of the sides.

$$d_1^2 + d_2^2 = 2\left(a^2 + b^2\right)$$

Sides 7 and 9, one diagonal 8

$$64 + d^2 = 2(49 + 81) = 260$$

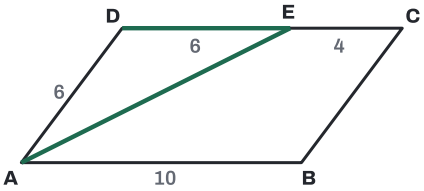
$$d^2 = 196$$

Other diagonal $= 14$ cm

Trap: $d_1^2 + d_2^2 = a^2 + b^2$: $\sqrt{66}$.

4 A bisector cuts off an isosceles triangle

$\angle DEA = \angle EAB$ (alternate) $= \angle DAE$, so $DE = AD$.



$AB = 10$, $AD = 6$; bisector of $\angle A$ meets CD at E

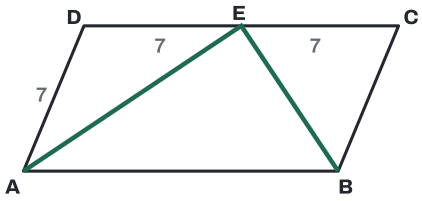
$DE = AD = 6$

$CE = 10 - 6 = 4$ **cm**

Trap: E taken as the midpoint: 5.

5 Two bisectors meet on the far side

Both cut-off triangles are isosceles, so $CD = 2\,AD$. The bisectors of neighbours always meet at 90°.



$AD = 7$; bisectors of $\angle A$, $\angle B$ meet on CD

$DE = 7$, $EC = 7$, so $AB = CD = 14$

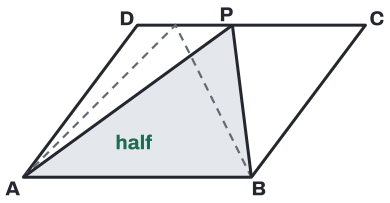
Perimeter $= 2(14 + 7) = 42$ cm

Trap: A rhombus of side 7: 28.

Areas and rectangles

1 The tip on the opposite side: always half

A triangle on one side, with its tip anywhere on the opposite side, has the same base and height: it is **half** the parallelogram.



E on BC, $BE : EC = 1 : 2$; $\triangle ABE = 10$

$\triangle ABC = \frac{1}{2}\, ABCD$

$\triangle ABE = \frac{1}{3}\triangle ABC = \frac{1}{6}\, ABCD$

$ABCD = 60$

$\triangle AED = \frac{1}{2} \times 60 = 30$ **sq cm**

Trap: $\triangle AED = 2 \times \triangle ABE$ from the $1 : 2$: 20.

2 Any point inside

The heights of $\triangle PAB$ and $\triangle PCD$ add to the full height.

Area 96; $\triangle PAB = 30$

$\triangle PAB + \triangle PCD = \frac{96}{2} = 48$

$\triangle PCD = 18$ **sq cm**

Trap: Taken equal to $\triangle PAB$: 30.

3 Area $= ab \sin\theta$

An obtuse angle θ gives the same area as its $180^\circ - \theta$ neighbour.

Sides 8 and 12, angle 150°

Height $= 8 \sin 30^\circ = 4$

Area $= 12 \times 4 = 48$ **sq cm**

Trap: $\sin 150^\circ$ read as $\frac{\sqrt{3}}{2}$: $48\sqrt{3}$.

4 Quarters

The diagonals make four equal triangles. Corner to the midpoint of a far side: also a quarter.

Area 84; diagonals meet at O

$\triangle AOB = \frac{84}{4} = 21$

E the midpoint of BC: $\triangle ABE = 21$ too

Trap: $\triangle ABE$ taken as half: 42.

5 Fold a corner onto the opposite corner

The crease is the perpendicular bisector of the diagonal: crease $=$ diagonal $\times \frac{\text{short}}{\text{long}}$.

Sheet 16×12; A folds onto C

$AC = \sqrt{256 + 144} = 20$

Half-crease $= 10 \times \frac{12}{16} = 7.5$

Crease $= 15$ **cm**

Trap: The crease taken as BC: 12.

6 Before you answer

Is it the longer or the shorter diagonal they ask? · A slant side is never the height. · "Any point" is not the midpoint. · Square: side $\times \sqrt{2}$ = diagonal, area $= \frac{d^2}{2}$. · Areas of similar figures go as the square of the sides.

Rhombus and square

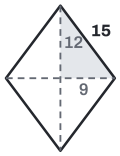
1 Rhombus: four right triangles

The diagonals cut each other in half at 90°. Each quarter is a right triangle with legs $\frac{d_1}{2}$, $\frac{d_2}{2}$ and hypotenuse a.

$d_1^2 + d_2^2 = 4a^2$

$(d_1 + d_2)^2 = 4a^2 + 4 \times \text{Area}$

$(d_1 - d_2)^2 = 4a^2 - 4 \times \text{Area}$



Side 25; the diagonals add to 62

$62^2 = 4 \times 25^2 + 4 \times \text{Area}$

$3844 = 2500 + 4 \times \text{Area}$

Area $= \frac{1344}{4} = 336$ **sq cm (diagonals** 48 **and** 14**)**

Trap: Forgetting the half: $d_1 d_2 = 672$ given as the area.

2 Side from the diagonals

Halve both diagonals first.

Diagonals 18 and 24

Halves 9 and 12

Side $= \sqrt{81 + 144} = 15$

Perimeter $= 60$ **cm**

Trap: Side $= \frac{18 + 24}{2} = 21$: 84.

3 The 60° rhombus

The short diagonal makes two equilateral triangles: it equals the side. The long one is $\sqrt{3}$ times it.

Angle 60°, short diagonal 8

Side $= 8$; long diagonal $= 8\sqrt{3}$

Area $= \frac{1}{2} \times 8 \times 8\sqrt{3} = 32\sqrt{3}$ **sq cm**

Trap: Long diagonal taken as 16: 64.

4 A circle inside a rhombus

It touches two opposite sides, so the height of the rhombus is the diameter. Side $= \frac{h}{\sin\theta}$.

Radius 6, angle 30°

Height $= 12$

Side $= \frac{12}{\sin 30^\circ} = 24$

Perimeter $= 96$ **cm**

Trap: Side $=$ diameter: 48.

5 Square plus an equilateral triangle

$AD = AB = AE$, so $\triangle ADE$ is isosceles.

$\triangle ABE$ equilateral, inside square $ABCD$

$\angle DAE = 90^\circ - 60^\circ = 30^\circ$

$\angle ADE = \frac{180^\circ - 30^\circ}{2} = 75^\circ$

$\angle EDC = 90^\circ - 75^\circ = 15^\circ$

Trap: The 30° at A read off: 30°.

6 Four tiles round a hole

Four equal $a \times b$ tiles round a square gap: the big side is $a + b$, the gap's side is $a - b$.

Trap: $\frac{169 - 49}{4} = 30$ guessed as 5×6: 6.

Big square 169, gap 49

$a + b = 13$, $a - b = 7$

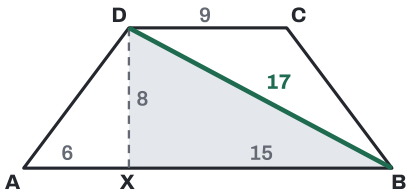
$a = 10$, $b = 3$

Longer side $= 10$ **cm**

Trapezium

1 Isosceles trapezium: $BX \times h$

Drop the height from D to X. The two overhangs are equal, so BX is the average of the parallel sides and the area is $BX \times h$. The diagonal BD is the hypotenuse on BX and h.



$AB = 21 \parallel DC = 9$, diagonal $BD = 17$

$BX = \dfrac{21+9}{2} = 15$

$h = \sqrt{17^2 - 15^2} = 8$

Area $= 15 \times 8 = 120$ **sq cm**

Trap: The diagonal as the height: $15 \times 17 = 255$.

2 Overhang is half the difference

Parallel sides 10 and 18, base angles 60°

Overhang $= \dfrac{18-10}{2} = 4$

$h = 4 \tan 60^\circ = 4\sqrt{3}$

Area $= 14 \times 4\sqrt{3} = 56\sqrt{3}$ **sq cm**

Trap: Overhang 8: $112\sqrt{3}$.

3 Perpendicular diagonals

In an isosceles trapezium they cut off two right isosceles triangles, so $h = \dfrac{a+b}{2}$ and the area is h^2.

Diagonals at 90°, height 9

$\dfrac{a+b}{2} = 9$

Area $= 9 \times 9 = 81$ **sq cm**

Trap: $\dfrac{1}{2}h^2$: 40.5.

4 The rung in a ratio $p : q$

Cutting each leg $p : q$ from the top, the weights cross over.

$$EF = \frac{q \cdot \text{top} + p \cdot \text{bottom}}{p+q}$$

Top 20, bottom 50, legs cut $1 : 2$

$EF = \dfrac{2 \times 20 + 1 \times 50}{3}$

$EF = 30$ **cm (midpoints: the average)**

Trap: Weights not crossed: 40.

5 Through the diagonals' crossing

The parallel line through the crossing is the harmonic mean, not the average.

$$\ell = \frac{2ab}{a+b}$$

Parallel sides 12 and 24

$\ell = \dfrac{2 \times 12 \times 24}{36}$

$\ell = 16$ **cm**

Trap: The midline: 18.

6 Four triangles from the diagonals

The two side triangles are equal, each $\sqrt{T_1 T_2}$.

$$\text{Area} = \left(\sqrt{T_1} + \sqrt{T_2}\right)^2$$

Triangles on the parallel sides: 9 and 25

Side triangles $= \sqrt{225} = 15$ each

Area $= 9 + 25 + 30 = 64$ **sq cm**

Trap: Side triangles as $\dfrac{9+25}{2} = 17$: 68.

7 Legs extended to a point

$\triangle ECD \sim \triangle EAB$: areas go as the square of $CD : AB$.

$AB = 10 \parallel CD = 6$; $\triangle ECD = 27$

$\triangle EAB = 27 \times \dfrac{100}{36} = 75$

Trapezium $= 75 - 27 = 48$ **sq cm**

Trap: Ratio not squared: $27 \times \dfrac{10}{6} - 27 = 18$.