Points and distance

1 Distance is Pythagoras

The horizontal gap and the vertical gap are the legs of a right triangle. The distance is its hypotenuse, so the order of the points never matters.

Quadrants	I $(+,+)$, II $(-,+)$, III $(-,-)$, IV $(+,-)$, counted anticlockwise
On an axis	On the y-axis $x = 0$, on the x-axis $y = 0$: the hidden equation
Distance	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Midpoint	The average: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

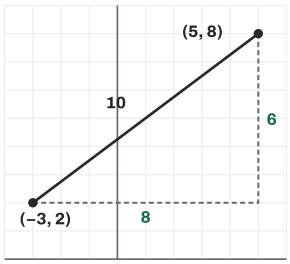
From $(-3, 2)$ to $(5, 8)$

Gaps: $5 - (-3) = 8$ and $8 - 2 = 6$

$(8, 6) = 2 \times (4, 3)$

$d = 2 \times 5$

$d = 10$



Trap: Adding the two gaps: $8 + 6 = 14$.

2 On an axis: put $x = 0$

Square both distances. The y^2 terms cancel and one simple equation is left.

On the y-axis, equally far from $(5, 6)$ and $(-1, 2)$

Point $(0, y)$

$25 + (y - 6)^2 = 1 + (y - 2)^2$

$61 - 12y = 5 - 4y$

$56 = 8y$

$(0, 7)$

Trap: The midpoint $(2, 4)$: equally far, but not on the axis.

3 Cross the ratio

If $AP : PB = m : n$, the m goes with B, one coordinate at a time: $P = (mB + nA) \div (m + n)$.

$A(2, -1)$, $B(12, 4)$, $AP : PB = 3 : 2$

$x = \frac{3 \cdot 12 + 2 \cdot 2}{5} = \frac{40}{5} = 8$

$y = \frac{3 \cdot 4 + 2 \cdot (-1)}{5} = \frac{10}{5} = 2$

$P = (8, 2)$

Trap: The 3 put with A: $(6, 1)$.

4 Where it crosses an axis

On the x-axis $y = 0$, so the axis cuts the segment in the ratio of the two heights.

From $(3, -4)$ to $(9, 8)$

Heights 4 and 8: ratio $1 : 2$

$x = \frac{1 \cdot 9 + 2 \cdot 3}{3} = \frac{15}{3} = 5$

Crosses at $(5, 0)$

Trap: The ratio the wrong way round: $(7, 0)$.

5 The centroid is an average

$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$. It sits two thirds of the way down each median.

$A(-4, 9)$; midpoint of BC is $D(5, 0)$

A to D: $(9, -9)$

Two thirds of it: $(6, -6)$

$G = (-4 + 6,\ 9 - 6)$

$G = (2, 3)$

Trap: One third from A: $(-1, 6)$.

Parallelograms and area

1 Area three ways

Look at the corners before you reach for a formula.

Two corners level	$\frac{1}{2} \times \text{base} \times \text{height}$: $(-2, 3)$, $(6, 3)$, $(1, -3)$ give $\frac{1}{2} \times 8 \times 6 = 24$
A tilted triangle	The box round it, minus the three right-angled corner triangles
Any polygon	Shoelace, corners in order: $\frac{1}{2}\lvert\sum(x_i y_{i+1} - x_{i+1} y_i)\rvert$
On one line	Three collinear points make a triangle of area 0

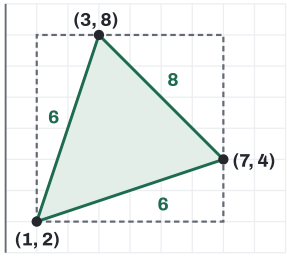
Corners $(1, 2)$, $(7, 4)$, $(3, 8)$

Box: $6 \times 6 = 36$

Corners: $\frac{1}{2} \cdot 6 \cdot 2 = 6$, $\frac{1}{2} \cdot 4 \cdot 4 = 8$, $\frac{1}{2} \cdot 2 \cdot 6 = 6$

$36 - (6 + 8 + 6)$

Area $= 16$



Trap: Half the box: 18.

2 Opposite corners add up

The diagonals share a midpoint, so $A + C = B + D$. Three corners with no order give three possible fourth corners.

$A(1, 2)$, $B(5, 3)$, $C(2, 6)$

$B + C - A = (6, 7)$

$A + C - B = (-2, 5)$

$A + B - C = (4, -1)$

In the first quadrant: $(6, 7)$

Trap: Assuming the order $ABCD$: $(-2, 5)$.

3 Midpoints back to the corners

A vertex and the three midpoints of the sides make a parallelogram $AFDE$.

Midpoints $D(3, 2)$ of BC, $E(5, 6)$ of CA, $F(1, 4)$ of AB

$A + D = E + F$

$A = (5 + 1 - 3,\ 6 + 4 - 2)$

$A = (3, 8)$

Trap: $D + E - F = (7, 4)$: that is C.

4 Shoelace

List the corners in order round the shape and repeat the first. Cross-multiply down each pair.

$(1, 1)$, $(8, 2)$, $(6, 7)$, $(2, 6)$

$1 \cdot 2 - 8 \cdot 1 = -6,\ 8 \cdot 7 - 6 \cdot 2 = 44$

$6 \cdot 6 - 2 \cdot 7 = 22,\ 2 \cdot 1 - 1 \cdot 6 = -4$

$\frac{1}{2}(-6 + 44 + 22 - 4)$

Area $= 28$

Trap: No half: 56.

5 Collinear: one slope

Every pair of points on a line gives the same slope.

$(k, 4)$, $(2, 10)$, $(6, 2)$ on a line

Slope $\frac{2 - 10}{6 - 2} = -2$

$\frac{4 - 10}{k - 2} = -2$

$k - 2 = 3$

$k = 5$

Trap: Slope as run over rise, $-\frac{1}{2}$: $k = 14$.

Equations of a line

1 Every form of a line

Slope	$m = \dfrac{y_2 - y_1}{x_2 - x_1} = \tan\theta$, the angle from the $+x$ axis
$ax + by = c$	Slope $-\dfrac{a}{b}$; the constant plays no part
From a slope	$y = mx + c$ (c with its sign), or $y - y_1 = m(x - x_1)$ through a point
Intercepts	$\dfrac{x}{a} + \dfrac{y}{b} = 1$; with the axes it makes a triangle of area $\dfrac{1}{2}\lvert ab\rvert$
Parallel, perpendicular	Keep a, b: $ax + by = k$; or swap them, flip one sign: $bx - ay = k$. The point gives k

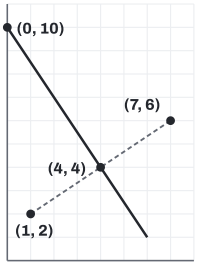
Equally far from $(1, 2)$ and $(7, 6)$

Midpoint $(4, 4)$; slope $\dfrac{4}{6} = \dfrac{2}{3}$, so $-\dfrac{3}{2}$

$y - 4 = -\dfrac{3}{2}(x - 4)$

At $x = 0$: $y = 4 + 6$

Meets the y-axis at $(0, 10)$



Trap: Only the sign flipped, $-\dfrac{2}{3}$: $\left(0, \dfrac{20}{3}\right)$.

2 Slopes multiply to -1

A right angle at B makes the opposite side a diameter of the circle through all three points.

Circle through $A(1, 1), B(5, 3), C(3, 7)$

$m_{AB} = \dfrac{2}{4} = \dfrac{1}{2}, m_{BC} = \dfrac{4}{-2} = -2$

Product -1: right angle at B, so AC is a diameter

Centre $(2, 4)$

Trap: The centroid $\left(3, \dfrac{11}{3}\right)$.

3 Intercepts keep their sign

A triangle on the y-axis has its base between the two y-intercepts.

$2x + y = 10$, $x - y = 2$ and the y-axis

y-intercepts 10 and -2: base 12

Add: $3x = 12$; they meet at $(4, 2)$, height 4

Area $= \dfrac{1}{2} \times 12 \times 4 = 24$

Trap: -2 read as 2: base 8, area 16.

4 Intercept form counts lines

Put the point into $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and factorise.

Through $(3, 2)$, intercepts a, b positive whole numbers

$\dfrac{3}{a} + \dfrac{2}{b} = 1$, so $2a + 3b = ab$

$(a - 3)(b - 2) = 6 = 1 \cdot 6 = 2 \cdot 3 = 3 \cdot 2 = 6 \cdot 1$

4 lines

Trap: Counting the negative pairs too: 8.

5 The gap between parallel lines

Match a and b first. Gap $= \dfrac{\lvert c_1 - c_2\rvert}{\sqrt{a^2 + b^2}}$.

$3x + 4y = 6$ and $6x + 8y = 37$

Halve: $3x + 4y = 18.5$

$\dfrac{\lvert 18.5 - 6\rvert}{\sqrt{9 + 16}} = \dfrac{12.5}{5}$

Gap $= 2.5$

Trap: Not scaled first: $\dfrac{37 - 6}{5} = 6.2$.

Distance, mirrors, lattice points

1 The nearest point on a line

The shortest way to a line meets it at a right angle. The distance from (x_0, y_0) to $ax + by = c$ is $\dfrac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}$.

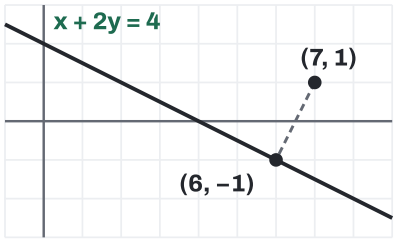
Well $(7, 1)$, road $x + 2y = 4$

Perpendicular: $2x - y = 2 \cdot 7 - 1 = 13$

$x + 2(2x - 13) = 4$, so $x = 6, y = -1$

Nearest point $(6, -1)$, distance $\sqrt{1 + 4} = \sqrt{5}$

Check: $\dfrac{|7 + 2 - 4|}{\sqrt{5}} = \dfrac{5}{\sqrt{5}} = \sqrt{5}$



Trap: Dropping straight down to $(7, -1.5)$: 2.5, longer than $\sqrt{5}$.

2 Reflect, then go straight

To touch a line on the way from A to B, mirror B in the line. The straight path to the image is the shortest.

$(1, 6)$ to the x-axis, then to $(9, 2)$

Mirror the end: $(9, -2)$

Slope $\dfrac{-2 - 6}{9 - 1} = -1$

$y = 0$ at $x = 1 + 6 = 7$

Touch at $(7, 0)$; path $\sqrt{8^2 + 8^2} = 8\sqrt{2}$

Trap: Halfway: $(5, 0)$.

3 A square from one side

Turn a step (p, q) a quarter turn anticlockwise: $(-q, p)$.

Square $ABCD$, $A(1, 2)$, $B(5, 5)$, C and D above AB

Step $A \to B$: $(4, 3)$

Turned: $(-3, 4)$

$D = A + (-3, 4)$, $C = B + (-3, 4) = (2, 9)$

$D = (-2, 6)$

Trap: $(-4, 3)$, only a sign changed: $(-3, 5)$.

4 Lattice points on a segment

Steps p across and q up: $\gcd(p, q)$ equal hops, so $\gcd(p, q) + 1$ points, ends included.

From $(-3, 2)$ to $(21, 20)$

Steps 24 and 18

$\gcd = 6$: hops of $(4, 3)$

$6 + 1 = 7$ points

Trap: One for every whole x: 25.

5 Pick's theorem

A polygon with lattice corners: area $= I + \dfrac{B}{2} - 1$, with I inside and B on the boundary.

$(0, 0)$, $(12, 0)$, $(0, 8)$

Area $\dfrac{1}{2} \times 12 \times 8 = 48$

$B = 12 + 8 + \gcd(12, 8) = 24$

$I = 48 - 12 + 1$

37 points inside

Trap: Counting the boundary too: 61.

6 Before you answer

Ratio crossed the right way? · Intercepts with their signs? · Perpendicular: fraction turned over and sign flipped? · The half on the shoelace? · Every possible fourth corner tried?