Arithmetic progressions

1 An AP is jumps of d

From term p to term q is $q - p$ jumps. Count jumps, not terms. No equations for two given terms.

nth term	$a_n = a_1 + (n-1)d$, and $a_q = a_p + (q-p)d$
Number of terms	$n = \dfrac{\text{last} - \text{first}}{d} + 1$
Sum of n terms	$S_n = \dfrac{n}{2}(\text{first} + \text{last}) = \dfrac{n}{2}\big(2a_1 + (n-1)d\big)$

Month 4: 230 books; month 11: 370

7 jumps $= 140$, so $d = 20$

Month $11 \to 20$: 9 more jumps

$370 + 9 \times 20 = 550$

Trap: 10 jumps from month 11: 570.

2 The middle is the average

a, b, c in AP $\iff 2b = a + c$.

$x + 2,\ 3x - 1,\ 4x + 3$ in AP

$2(3x - 1) = (x + 2) + (4x + 3)$

$6x - 2 = 5x + 5$, so $x = 7$

Third term $= 4(7) + 3 = 31$

Trap: $b = a + c$, no 2: $x = -3$, gives -9.

3 The sum is pairs

First + last = second + second-last = ...

$17, 21, 25, \ldots, 213$

$n = \dfrac{213 - 17}{4} + 1 = 50$

25 pairs, each $17 + 213 = 230$

Sum $= 25 \times 230 = 5{,}750$

Trap: 49 terms: $5{,}635$.

4 The central term is the average

Odd count: $S_n = n \times$ middle term. No a or d needed.

$a_8 = 23$; find S_{15}

Middle of 1 to 15 is term 8

$S_{15} = 15 \times 23 = 345$

Trap: $\dfrac{15}{2} \times 23 = 172.5$.

5 Assume terms around the centre

Odd: $m - d,\ m,\ m + d$.

Even: $m - 3e,\ m - e,\ m + e,\ m + 3e$ (step $2e$).

Four terms add to 36; middle two multiply to 77

$m = \dfrac{36}{4} = 9$

$(9 - e)(9 + e) = 77$, so $e = 2$

Largest $= 9 + 3e = 15$

Trap: Step e: $9 + 2e = 13$.

More AP tricks

1 $S_n = An^2 + Bn$ is always an AP

Read it off: $d = 2A$ and $a_1 = A + B$. Any term is $S_n - S_{n-1}$. Terms p to q add to $S_q - S_{p-1}$.

$S_n = 3n^2 + 5n$: here $d = 6$, $a_1 = 8$

Terms 10 to 20: $S_{20} - S_9$

$S_{20} = 1200 + 100 = 1300$

$S_9 = 243 + 45 = 288$

$1300 - 288 = 1{,}012$

Trap: $S_{20} - S_{10}$ drops term 10: 950.

2 A block's average is its middle term

Average of 7 days = the 4th of those days.

Week 1 averages 20; week 2 averages 34

Day $4 = 20$, day $11 = 34$

7 jumps $= 14$, so $d = 2$

Day $20 = 34 + 9 \times 2 = 52$

Trap: $\frac{14}{14} = 1$ a day: 43.

3 Same size, opposite sign

$|a_p| = |a_q|$, $d \neq 0$: term $\frac{p+q}{2}$ is 0, and $S_{p+q-1} = 0$.

$a_1 = 18$, $|a_6| = |a_{14}|$

$a_{10} = 0$: 9 jumps from 18 to 0

$d = -2$

$a_{20} = 0 + 10 \times (-2) = -20$

Trap: 18 mirrors to a_{19}, not a_{20}: -18.

4 Ratio of terms, ratio of sums

$S_{2k-1} = (2k-1)\,a_k$. So the kth terms go with $2k - 1$ terms.

$a_n : b_n = (4n-1) : (6n+1)$; sums of 11 terms?

11 terms average to the 6th term

$a_6 : b_6 = 23 : 37$

Trap: $n = 11$ in the term ratio: $43 : 67$.
Given sums, want terms: put $n = 2k - 1$ into the sum ratio.

5 Common terms step by the LCM

Find the first common term. Then step by the LCM of the two steps.

Bus A: $5, 11, 17, \ldots$; bus B: $1, 9, 17, \ldots$

First common $= 17$

Step $= \mathrm{LCM}(6, 8) = 24$

10**th common** $= 17 + 9 \times 24 = 233$

Trap: Step $6 \times 8 = 48$: 449.

Geometric progressions

1 A GP is jumps of $\times r$

nth term	$a_n = a_1 r^{n-1}$, and $a_q = a_p\, r^{q-p}$
Sum of n terms	$S_n = \dfrac{a_1(r^n - 1)}{r - 1} = \dfrac{\text{last} \times r - \text{first}}{r - 1}$
Infinite sum	$S_\infty = \dfrac{a_1}{1 - r}$, only for $-1 < r < 1$

Month 2: 6 thousand; month 5: 162 thousand

$r^3 = \dfrac{162}{6} = 27$, so $r = 3$

Month $7 = 162 \times 3^2 = 1{,}458$ thousand

Trap: $27 \div 3 = 9$ as the ratio: 13,122.

2 $b^2 = ac$

The middle squared is the product of its neighbours.

$x,\ x+3,\ x+9$ in GP

$(x+3)^2 = x(x+9)$

$6x + 9 = 9x$: $x = 3$, GP $3, 6, 12, \ldots$

6th term $= 3 \times 2^5 = 96$

Trap: $3 \times 2^6 = 192$.

3 Divide one equation by the other

Shift by k terms: the ratio is r^k.

Terms 1–3 add to 21; terms 4–6 to -168

$r^3 = \dfrac{-168}{21} = -8$, so $r = -2$

$a(1 - 2 + 4) = 21$, so $a = 7$

$a_5 = 7 \times (-2)^4 = 112$

Trap: Dropping the sign, $r = 2$: 48.

4 Assume $a/r,\ a,\ ar$

The product is a^3. The pair products add to $a\times$ the sum.

Product 1728, sum 38: $a^3 = 1728$, $a = 12$

Sum of squares $= 38^2 - 2 \times 12 \times 38$

$1444 - 912 = 532$

Trap: $38^2 - 2 \times 12^2 = 1{,}156$.

5 The GP sum

Doubling: the total is the next term minus the first.

₹5 on day 1, doubled every day, 10 days

$S_{10} = \dfrac{5(2^{10} - 1)}{2 - 1}$

$5 \times 1023 =$ ₹5,115

Trap: $5 \times 2^{10} =$ ₹5,120.

6 The endless GP

The first k terms are $S_\infty(1 - r^k)$.

$S_\infty = 96$; first two terms add to 90

$96(1 - r^2) = 90$, so $r^2 = \dfrac{1}{16}$

Positive terms: $r = \dfrac{1}{4}$

$a = 96 \times \dfrac{3}{4} = 72$

Trap: Using $96(1 - r) = 90$ gives $a = 90$.

7 Shift the recurrence

$x_{n+1} = px_n + q$: $c = \dfrac{q}{1 - p}$ stays put, and $x_n - c$ is a GP.

76 L; each hour halve, then add 6 L

$c = \dfrac{6}{1/2} = 12$; gap $76 - 12 = 64$

6 hours: $12 + \dfrac{64}{2^6} = 13$ L

Trap: 7 halvings: 12.5 L.

Special sums and sigma

1 Sums that start from 1

$\sum k$	$\frac{n(n+1)}{2}$; to 10: 55
$\sum k^2$	$\frac{n(n+1)(2n+1)}{6}$; to 10: 385
$\sum k^3$	$\left(\frac{n(n+1)}{2}\right)^2$; to 10: 3025
Odd, even	$1+3+\cdots+(2n-1)=n^2$; $2+4+\cdots+2n=n(n+1)$
Rules	$\sum(f\pm g)=\sum f\pm\sum g$; $\sum_{k=1}^{n} c=nc$; open a product first

Starts late? Add the missing start, use the formula, then take it back.

$11^2+12^2+\cdots+20^2$

$\sum_1^{20} k^2=\frac{20\times21\times41}{6}=2870$

Take back $\sum_1^{10} k^2=385$

$2870-385=2{,}485$

Trap: Taking back up to 11^2: 2,364.

2 Open the bracket

Expand, then one sigma per piece.

$1^2+3^2+\cdots+23^2=\sum_{k=1}^{12}(2k-1)^2$

$=4\sum k^2-4\sum k+\sum 1$

$=4(650)-4(78)+12$

$=2600-312+12=2{,}300$

Trap: $\sum 1$ as 1: 2,289.

3 Squares in pairs

$a^2-b^2=(a+b)(a-b)$. Here $a-b=1$.

$20^2-19^2+18^2-17^2+\cdots+2^2-1^2$

$=(20+19)+(18+17)+\cdots+(2+1)$

$=1+2+\cdots+20=210$

Trap: Each pair as 1: 10.

4 Split the fraction

$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$. The middle cancels.

n terms add to $1-\frac{1}{n+1}=\frac{n}{n+1}$

$\frac{n}{n+1}>0.95\Rightarrow n>19$

First past 0.95: $n=20$

Trap: $n=19$ gives exactly 0.95.

5 AM, GM, HM of two numbers

$\text{AM}\geq\text{GM}\geq\text{HM}$ and $\text{GM}^2=\text{AM}\times\text{HM}$.

$\text{AM}=25$, $\text{GM}=20$

$\text{HM}=\frac{20^2}{25}$

HM $=16$ (the numbers are 40 and 10)

Trap: Equal gaps: 15.

6 Before you answer

Jumps = terms – 1. · Odd count: the middle term is the average. · Formulas start from 1: give, then take back. · GP terms divide; AP terms subtract.