The wavy curve

1 Sign chart in four moves

Bring everything to one side. Factorise. Pin each critical point. The far right takes the sign of the product of the coefficients of x, then the sign alternates leftwards.

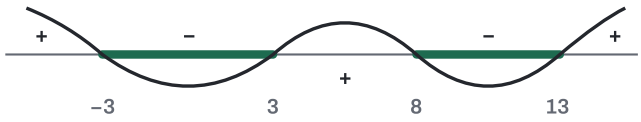
A negative	Multiplying or dividing by it flips the sign. A factor $(a - x)$ is $-(x - a)$
An unknown	Never cancel it and never cross-multiply it: it may be negative
$\leq$ **or** $\geq$	Close the brackets at the numerator's zeros, never at the denominator's

$(x^2 - 9)(x - 8)(13 - x) > 0$

Flip: $(x + 3)(x - 3)(x - 8)(x - 13) < 0$

Negative regions: $(-3, 3)$ and $(8, 13)$

Integers: $5 + 4 = 9$



Trap: Keeping $(13 - x)$ as it is: only 4 to 7, so 4.

2 Squaring a range

If the range crosses 0, the smallest square is 0. The end further from 0 sets the top.

$-5 < 2x - 1 < 7$

$-4 < 2x < 8$

$-2 < x < 4$

$0 \leq x^2 < 16$

16 **integer values,** 0 **to** 15

Trap: $(-2)^2$ to 4^2: 5 to 15, so 11.

3 Push the other one to its edge

To make one variable as big as possible, give the other its smallest allowed value.

$12s + 30j \leq 600, j \geq 5, s \geq 2j$

$j = 5$: $12s \leq 450$

$s \leq 37.5$

Most samosas $= 37$

Trap: Ignoring the juice: $600 \div 12 = 50$.

4 Even powers bounce

A square or a fourth power never changes sign. Its zero joins only when "$=$" is allowed.

$(x + 4)^2(x - 1)(x - 7)^4(x - 9) \leq 0$

Sign from $(x - 1)(x - 9) \leq 0$: $[1, 9]$

Add the bounce zero $x = -4$

$9 + 1 = 10$ **integers**

Trap: Missing -4: 9.

5 Fractions: never cross-multiply

The denominator may be negative. Move everything to one side.

$$\frac{x + 2}{x - 5} \leq \frac{1}{2}$$

$$\frac{2(x + 2) - (x - 5)}{2(x - 5)} \leq 0$$

$$\frac{x + 9}{2(x - 5)} \leq 0$$

$-9 \leq x < 5$**:** 14 **integers**

Trap: Cross-multiplying: $x \leq -9$, endless.

Quadratics, cubics, AM–GM

1 Always one sign: $D < 0$

A quadratic with $D = b^2 - 4ac < 0$ never meets the axis, so it keeps the sign of a. It can be dropped from a product, and "for every real x" means $D < 0$.

$a > 0, D < 0$	Positive for every x: $x^2 + x + 1$
$a < 0, D < 0$	Negative for every x: $-x^2 + x - 2$
$D = 0$	Touches 0 once, so "> 0 for every x" fails

$x^2 + (k-2)x + 16 > 0$ for every real x

$D = (k-2)^2 - 64 < 0$

$-8 < k - 2 < 8$

$-6 < k < 10$

15 **integers** k

Trap: $D \leq 0$ lets it touch 0: 17.

2 Cubic: one root by trial

Coefficients add to 0, so $x = 1$ is a root. Sum and product give the other two.

$x^3 - 14x^2 + 49x - 36 < 0$

$1 - 14 + 49 - 36 = 0$, so $x = 1$

Others: sum 13, product 36: 4 and 9

$(x-1)(x-4)(x-9) < 0$

$x < 1$ or $4 < x < 9$

Positive integers 5 **to** 8**:** 4

Trap: Starting from below: $(1, 4)$, so 2.

3 Equal parts, biggest product

A fixed sum gives the biggest product when the parts are equal. Make the weighted parts equal, not x and y.

Pen on a wall, split in two: $3x + y = 60$

$$3x \cdot y \leq \left(\frac{60}{2}\right)^2 = 900$$

$3x = y = 30$, so $x = 10$

Area $xy = \dfrac{900}{3} = 300$ **m²**

Trap: $x = y = 15$: 225 m².

4 $x + \dfrac{c}{x}$ skips a gap

For $x > 0$ it is at least $2\sqrt{c}$. For $x < 0$ it is at most $-2\sqrt{c}$.

$x + \dfrac{9}{x} = k$ has no real root

$x + \dfrac{9}{x} \geq 6$ or $x + \dfrac{9}{x} \leq -6$

Impossible: $-6 < k < 6$

11 **integers,** -5 **to** 5

Trap: Forgetting $x < 0$: "every $k < 6$".

5 Count the pairs

Fix the smaller variable, count the other, then add.

$40n + 15p \leq 300, n \geq 2, p > n$

$n = 2$: $p = 3$ to 14, so 12

$n = 3$: 9; $n = 4$: 5; $n = 5$: 1

$12 + 9 + 5 + 1 = 27$

Trap: "More than" read as "at least": 31.

Modulus is distance

1 $|x-a|$ is how far x is from a

Read every modulus as a gap on the number line. No cases, no squaring.

$\|x-a\| = c$	Two points: $a-c$ and $a+c$
$\|x-a\| < c$	Between them: $a-c < x < a+c$
$\|x-a\| > c$	Outside: $x < a-c$ or $x > a+c$
$\|x+a\|$	Distance from $-a$
$\|px-q\|$	px is the point at that distance from q; divide by p at the end

$3 \le |2x-13| \le 9$

$2x$ is 3 to 9 away from 13

$2x \in [16, 22]$ or $2x \in [4, 10]$

$x \in [8, 11]$ or $x \in [2, 5]$

$4+4=8$ **integers**

Trap: Only the right-hand band: 4.

2 Twice as far

$|x-a| = m|x-b|$: one point inside the gap, one outside it.

$|x-3| = 2|x-9|$

Inside: $x-3 = 2(9-x)$, so $x = 7$

Outside: $x-3 = 2(x-9)$, so $x = 15$

Sum $= 7+15 = 22$

Trap: Stopping at the inside point: 7.

3 Nearer to one point

$|x-a| < |x-b|$: the side of the midpoint $\frac{a+b}{2}$ that holds a.

$|x-9| < |x+5|, -20 \le x \le 20$

$|x+5| = |x-(-5)|$

Midpoint $\frac{9+(-5)}{2} = 2$, so $x > 2$

3 **to** 20**:** 18 **integers**

Trap: Midpoint $\frac{9+5}{2} = 7$: 13.

4 Two moduli: the spare splits

$|x-a| + |x-b|$ equals the gap $b-a$ all the way between a and b. For $= k$: none below the gap, endless at the gap, two above it.

$|x+4| + |x-6| \le 16$

Gap 10, spare 6: 3 past each end

$-7 \le x \le 9$

17 **integers**

Trap: 6 past each end: 23.

5 Many points: the median

A sum of distances $\sum |x-a_i|$ is smallest at the median point.

Shops at $2, 5, 11, 14, 30$

Median 11

$9+6+0+3+19$

Least total $= 37$

Trap: At the mean 12.4: 38.4.

Modulus graphs and cases

1 Draw it from the corners

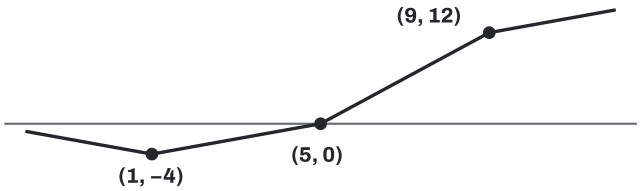
Work out the value at each critical point and join them with straight lines. Each end slope is the net coefficient of x with every modulus opened as it is. Solutions of $f(x) = k$ are the crossings of the line $y = k$.

$f(x) = |x-1| + |x-5| - |x-9|$

$f(1) = -4,\ f(5) = 0,\ f(9) = 12$

End slopes: $-1-1+1 = -1,\ 1+1-1 = 1$

Both ends rise: least value -4 at $x = 1$



Trap: The median rule needs every modulus added: at $x = 5$ it gives 0.

2 Keep a root only in its region

Split at the critical point and solve each case. A root counts only on its own side.

$x^2 + 5|x-1| = 19$

$x \geq 1$: $x^2 + 5x - 24 = 0$: 3 yes, -8 no

$x < 1$: $x^2 - 5x - 14 = 0$: -2 yes, 7 no

Sum $= 3 + (-2) = 1$

Trap: Keeping all four roots: 0.

3 Never cancel $|x| - x$

$|x| - x$ is 0 for $x \geq 0$ and $-2x$ for $x < 0$.

$(|x| - x)(x+9) \leq 0,\ -15 \leq x \leq 15$

$x \geq 0$: product 0, all 16 count

$x < 0$: need $x \leq -9$: -15 to -9, so 7

$16 + 7 = 23$

Trap: Cancelled as positive: 7.

4 The flipped hump

$|q(x)|$ flips the dip above the axis. The hump's height is $|q|$ at the vertex.

$|x^2 - 10x + 16| < 9$

Vertex $x = 5$: $|25 - 50 + 16| = 9$

$x^2 - 10x + 16 < 9$: $5 - \sqrt{18} < x < 5 + \sqrt{18}$

Integers 1 to 9, but not 5

8 integers

Trap: Counting the hump top $x = 5$: 9.

5 $|x| + |y| \leq k$ is a diamond

Its diagonals are $2k$ and $2k$, so the area is $2k^2$. Shifting it does not change the area.

$|x-3| + |y+2| \leq 6$

Diagonals 12 and 12

Area $= \frac{1}{2} \times 12 \times 12$

72 square units

Trap: A 12×12 square: 144.

6 Before you answer

Flipped every $(a - x)$? · Bounce points and "=" checked? · Denominator's zero left open? · Both sides of a modulus, inside and outside? · Each root in its own region?