The function machine

1 Look for what changes

The input goes wherever the function's own letter is. Changing x to $-x$ flips only the odd powers.

$f(x)+f(-x)$	$2\times$ the even-power part: the odd terms cancel
$f(x)-f(-x)$	$2\times$ the odd-power part: the even terms cancel

$f(x)=ax^5+bx^3+2x^2+3$, a and b unknown, $f(2)=40$

Even part at 2: $2\times4+3=11$

$f(2)+f(-2)=2\times11=22$

$f(-2)=22-40=-18$

Trap: Just flipping the sign: -40.

2 Which rule owns the input

Check the conditions first. A boundary value belongs to the rule with $\le$ or $\ge$.

x^2 if $x<0$; $3x+1$ if $0\le x<4$; $2x+10$ if $x\ge4$

$f(-2)=(-2)^2=4$

$4\ge4$: third rule

$f(f(-2))=2\times4+10=18$

Trap: 4 into $3x+1$: 13.

3 Composite: inside first

Work from the innermost bracket out, one number at a time.

$f(x)=x^2-3$, $g(x)=2x+1$; find $g(f(g(1)))$

$g(1)=3$

$f(3)=9-3=6$

$g(6)=13$

Trap: Wrong order, $f(g(f(1)))$: 6.

4 A line inside a line

$f=ax+b$ applied 3 times: $a^3x+b(a^2+a+1)$.

$f(f(f(x)))=8x+21$

$a^3=8\Rightarrow a=2$

$b(4+2+1)=21\Rightarrow b=3$

$f(2)=2\times2+3=7$

Trap: Constant as $b(a+1)$: $b=7$, $f(2)=11$.

5 Loops: find the cycle

Apply f until an output repeats. Then only the remainder counts.

$f(x)=\frac{1}{1-x}$; find $f^{50}(3)$

$3\to-\frac{1}{2}\to\frac{2}{3}\to3$: a cycle of 3

$50=3\times16+2$

$f^{50}(3)=f^2(3)=\frac{2}{3}$

Trap: One step early: $-\frac{1}{2}$.

6 Rules that loop

A piecewise rule or a recurrence can circle back too. Write the values until the start comes round again.

$f(n)=\frac{n}{2}$ if n is even, $n+5$ if odd; find $f^{100}(6)$

$6\to3\to8\to4\to2\to1\to6$: a cycle of 6

$100=6\times16+4$

$f^{100}(6)=f^4(6)=2$

$f(1)=4$, $f(2)=7$, $f(n+2)=f(n+1)-f(n)$

$4,7,3,-4,-7,-3$, then $4,7,\ldots$ again; every 6 sum to 0

$100=6\times16+4$

$f(1)+\cdots+f(100)=4+7+3-4=10$

Trap: A full cycle sums to 0, but 100 terms are not whole cycles: not 0.

Quadratic functions

1 Roots first, then k

Write the shape from what you are told, then one more value fixes k.

Zeros at α, β	$f(x) = k(x-\alpha)(x-\beta)$
$f \geq 0$ **(or** ≤ 0**) for all** x**, and** $f(\alpha) = 0$	A double root: $f(x) = k(x-\alpha)^2$
$f(0)$	The constant c: where the graph cuts the y-axis
$f(x) \pm f(-x)$	$2ax^2 + 2c$ or $2bx$

Zeros at -1 and 5; $f(0) = -10$

$f(x) = k(x+1)(x-5)$

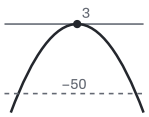
$f(0) = -5k = -10 \Rightarrow k = 2$

$f(3) = 2 \times 4 \times (-2) = -16$

Trap: Taking $k = 1$: -8.

2 Never above zero

Touches the axis at its only zero, so a double root.



$f \leq 0$ always, $f(3) = 0$, $f(0) = -18$

$f = k(x-3)^2$, $9k = -18$, $k = -2$

$f \geq -50$: $(x-3)^2 \leq 25$

$-2 \leq x \leq 8$

11 **integers**

Trap: $8 - (-2) = 10$.

3 The odd part alone

$f(x) - f(-x) = 2bx$. Halve it.

$f(x) - f(-x) = 8x$, $f(0) = 3$, $f(2) = 23$

$2b = 8 \Rightarrow b = 4$; $c = 3$

$4a + 8 + 3 = 23 \Rightarrow a = 3$

$f(-1) = 3 - 4 + 3 = 2$

Trap: $b = 8$, $a = 1$: -4.

4 One meeting point

Set them equal. One meeting point means $D = 0$.

$y = kx + 3$ $(k > 0)$ touches $y = x^2 + 3x + 7$

$x^2 + (3-k)x + 4 = 0$

$(3-k)^2 = 16 \Rightarrow k = 7$ or -1

$k > 0$**:** $k = 7$

Trap: The root $k > 0$ rules out: -1.

5 Sign from the graph

$a > 0$: negative between the roots, positive outside.
$a < 0$: the other way round.

$f = (x-2)(x-6)$; sign of $f(3) \cdot f(7)$?

3 is between the roots: $f(3) = -3$

7 is outside: $f(7) = 5$

$f(3) \cdot f(7) = -15 < 0$

Trap: Both outside the roots? Then positive. Check which side.

Domain and range

1 Domain: what x is allowed

Domain is the inputs that give a real output. Range is the outputs that really occur.

Square root	Inside ≥ 0
Denominator	$\neq 0$: its roots stay open, even under a root
A ratio of factors	Wavy curve: mark the roots, start $+$ on the right, flip at each one

Integers from -5 to 10 where $\sqrt{\dfrac{(x+2)(x-3)}{x-6}}$ is real

Roots $-2, 3, 6$; $+$ right of 6, then $-, +, -$

Domain $[-2, 3] \cup (6, \infty)$

$-2, \ldots, 3$ **and** $7, \ldots, 10$**:** $6 + 4 = 10$ **integers**

Trap: Letting $x = 6$ in: 11.

2 The value it never takes

$\frac{ax+b}{cx+d}$ misses only the output $\frac{a}{c}$. The missing input is $-\frac{d}{c}$.

$\dfrac{ax+5}{2x-3}$ never equals 4

$\frac{a}{2} = 4 \Rightarrow a = 8$

$f(2) = \dfrac{16+5}{4-3} = 21$

Trap: $a = 4$: $f(2) = 13$.

3 Split off the constant

Same x^2 and x terms, denominator with $D < 0$: write $k + \frac{m}{\text{denominator}}$.

$\dfrac{2x^2+4x+11}{x^2+2x+3} = 2 + \dfrac{5}{(x+1)^2+2}$

Denominator ≥ 2: top value $2 + \frac{5}{2} = 4.5$

Range $(2,\ 4.5]$

Integer values 3 **and** 4**: two**

Trap: Counting 2, which is never reached: three.

4 AM–GM for the top value

Divide top and bottom by the top. For $t > 0$, $t + \frac{m}{t} \geq 2\sqrt{m}$.

$\dfrac{x+1}{x^2+2x+10}$, $x > -1$; put $t = x + 1$

$\dfrac{t}{t^2+9} = \dfrac{1}{t + 9/t}$

$t + \frac{9}{t} \geq 6$, at $t = 3$

Largest value $= \frac{1}{6}$

Trap: Putting $x = 0$: $\frac{1}{10}$.

5 Range by the discriminant

Write $y = f(x)$ as a quadratic in x. A real x needs $D \geq 0$. Check $y = 0$ separately.

$y = \dfrac{x}{x^2+1}$: $yx^2 - x + y = 0$

$D = 1 - 4y^2 \geq 0 \Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2}$

$y = 0$ at $x = 0$

Range $[-\frac{1}{2}, \frac{1}{2}]$

Trap: Dividing by y without checking $y = 0$.

Functional equations and series

1 Read the rule, guess the function

$f(x+y)=f(x)\,f(y)$	$f(x)=a^x$
$f(xy)=f(x)+f(y)$	$f(x)=k\log x$, so $f(1)=0$
$f(x+y)=f(x)+f(y)$	$f(x)=kx$
$f(xy)=f(x)\,f(y)$	$f(x)=x^n$
$f(x)f(\frac{1}{x})=f(x)+f(\frac{1}{x})$	$f(x)=1\pm x^n$

$f(xy)=f(x)+f(y)$, $f(8)=6$; find $f(\frac{1}{16})$

$f(8)=3f(2)\Rightarrow f(2)=2$

$f(16)+f(\frac{1}{16})=f(1)=0$

$f(\frac{1}{16})=-4\times 2=-8$

Trap: Sign dropped: 8.

2 Swap x and $\frac{1}{x}$, then add

Put $\frac{1}{x}$ for x, then add the two equations.

$f(x)+3f(\frac{1}{x})=8x$

$f(\frac{1}{x})+3f(x)=\frac{8}{x}$

Add: $4[f(x)+f(\frac{1}{x})]=8(x+\frac{1}{x})$

$f(2)+f(\frac{1}{2})=2\times\frac{5}{2}=5$

Trap: $\frac{16}{4}=4$.

3 Pairs that add to 1

$\frac{a^x}{a^x+c}$: pair the inputs where $a^x\cdot a^{x'}=c^2$.

$f(x)=\frac{9^x}{9^x+9}$: $f(x)+f(2-x)=1$

$f(\frac{1}{10})+\cdots+f(\frac{19}{10})$: 9 pairs

Middle term $f(1)=\frac{9}{18}=\frac{1}{2}$

Sum $=9.5$

Trap: Middle counted as a pair: 10.

4 Build it step by step

No standard form? Grow the values one step at a time.

$f(x+y)=f(x)+f(y)+2xy$, $f(1)=4$

$f(2)=10$, $f(3)=18$, $f(4)=28$

Pattern x^2+3x

$f(5)=40$

Trap: Taking f linear: 20.

5 Flip it to telescope

Divide by $f(n)$: the reciprocals move by a constant.

$f(1)=\frac{1}{3}$, $f(n+1)=\frac{f(n)}{1+2f(n)}$

$\frac{1}{f(n+1)}=\frac{1}{f(n)}+2$

9 steps from $f(1)$: $3+2\times 9=21$

$f(10)=\frac{1}{21}$

Trap: 10 steps: $\frac{1}{23}$.

6 Greatest integer $[x]$ and fractional part $\{x\}$

$x=[x]+\{x\}$ with $0\le\{x\}<1$, and $[-2.2]=-3$. $[x]$ changes only at thresholds: count each run.

$[\log_2 1]+[\log_2 2]+\cdots+[\log_2 20]$

Values $0,1,2,3,4$ run for $1,2,4,8,5$ terms

$0+1\times 2+2\times 4+3\times 8+4\times 5$

$=54$

$2x+3[x]=26.4$

$5[x]+2\{x\}=26.4$ and $2\{x\}<2$

$[x]=5$, $\{x\}=0.7$

$x=5.7$, **so** $10x=57$

Trap: Last run cut to 4: 50. Treating $[x]$ as x: $10x=52.8$.