Same base

1 Every base in primes

Break each base into primes. Then compare one prime at a time: its power on the left must equal its power on the right.

Multiply, divide	$a^m \times a^n = a^{m+n},\ a^m \div a^n = a^{m-n}$
Power of a power	$(a^m)^n = a^{mn},\ (ab)^n = a^n\, b^n$
Zero, negative, root	$a^0 = 1,\ a^{-n} = \dfrac{1}{a^n},\ \sqrt[n]{a^m} = a^{m/n}$

$24^a \times 45^b = 2^9 \times 3^7 \times 5^2$

$24 = 2^3 \times 3,\ 45 = 3^2 \times 5$

Fives: $b = 2$. Twos: $3a = 9$, so $a = 3$

Threes check: $a + 2b = 3 + 4 = 7$

$a + b = 5$

Trap: Reading 24^a as only a twos: $a = 9$, so 11.

2 Roots are fractions

A cube root is a power of $\frac{1}{3}$. Write both sides in one base.

$\sqrt[3]{16^x} \times 4 = 2^{x+5}$

$2^{4x/3} \times 2^2 = 2^{x+5}$

$\dfrac{4x}{3} + 2 = x + 5$

$\dfrac{x}{3} = 3$

$x = 9$

Trap: Dropping the root: $4x + 2 = x + 5$ gives 1.

3 Compare at one power

Raise every root to the LCM of the root orders.

$\sqrt[3]{4},\ \sqrt[4]{6},\ \sqrt[6]{15}$

LCM of $3, 4, 6$ is 12

$4^4 = 256,\ 6^3 = 216,\ 15^2 = 225$

Largest: $\sqrt[3]{4}$

Trap: The biggest base wins: $\sqrt[6]{15}$.

4 Make tens

Pair each 2 with a 5. Only the leftover sets the digits.

$12^5 \times 5^{12} = 2^{10} \times 3^5 \times 5^{12}$

$= 10^{10} \times 3^5 \times 5^2$

$= 6075 \times 10^{10}$

$4 + 10 = 14$ **digits**

Trap: Forgetting the spare 5^2: 243 and ten zeros, 13.

5 Different bases: power zero

For integers, $2^m = 7^n$ only when $m = n = 0$. Both sides are 1.

$2^{p-2q} = 7^{q-3}$

$q - 3 = 0$ and $p - 2q = 0$

$q = 3,\ p = 6$

$p + q = 9$

Trap: "2 can never equal 7": no solution.

Hidden structure

1 Substitute $t = a^x$

Write every term as a power of one a^x: $4^x = (2^x)^2$ and $2^{x+1} = 2 \times 2^x$. The equation becomes a quadratic in t. Then turn each t back into x.

$4^x - 6 \times 2^{x+1} + 32 = 0$

$t = 2^x$: $t^2 - 12t + 32 = 0$

$(t - 4)(t - 8) = 0$

$2^x = 4$ or $2^x = 8$, so $x = 2$ or $x = 3$

Sum of $x = 5$

Trap: Adding the t values: 12.

2 t cannot be negative

$a^x > 0$ for every real x. A negative t gives no x.

$5^x - 5^{1-x} = 4$

$t - \frac{5}{t} = 4$

$t^2 - 4t - 5 = 0$: $t = 5$ or $t = -1$

Only $5^x = 5$: one solution, $x = 1$

Trap: Two roots of t, so "two solutions".

3 Take the smallest power out

Shifted powers of one base share the smallest one.

$5 \times 2^{x+1} - 3 \times 2^{x-1} = 136$

$2^{x-1}(5 \times 4 - 3) = 136$

$2^{x-1} = \frac{136}{17} = 8 = 2^3$

$x - 1 = 3$, **so** $x = 4$

Trap: Stopping at $2^{x-1} = 8$ and writing 3.

4 A common value

If $a^x = k$, then $a = k^{1/x}$. Multiplying bases adds the powers.

$2^x = 9^y = 6^z = k$

$2 = k^{1/x}$, $3 = k^{1/(2y)}$, $6 = k^{1/z}$

$2 \times 3 = 6$: $\frac{1}{x} + \frac{1}{2y} = \frac{1}{z}$

$\frac{z(x + 2y)}{xy} = 2$

Trap: Treating 9 as 3: 1.

5 Base equals exponent

a^b needs b to divide every prime's power.

$a^b = 2^{20} \times 3^{30}$, $a, b > 1$

b divides $\gcd(20, 30) = 10$

$b = 2, 5, 10$

3 pairs

Trap: Counting $b = 1$: 4.

6 Powers in a range

Find the first and last power inside. Count = last − first + 1.

Powers of 3 between 1000 and 10^6

$3^6 = 729$, $3^7 = 2187$

$3^{12} = 729^2 = 531441$, $3^{13} > 10^6$

$n = 7$ **to** 12**: 6 powers**

Trap: $12 - 7 = 5$.

A log is an exponent

1 The rules

$\log_b N$ asks: which power of b gives N? Every log rule is an exponent rule read backwards.

Meaning	$\log_b N = k \iff b^k = N$, with $b > 0, b \neq 1, N > 0$
Product, quotient	$\log mn = \log m + \log n$, $\log \dfrac{m}{n} = \log m - \log n$
Powers	$\log_{a^m} b^n = \dfrac{n}{m} \log_a b$
Change of base	$\log_a b = \dfrac{\log b}{\log a} = \dfrac{1}{\log_b a}$
Log in the power	$a^{\log_a b} = b$, $a^{\log_b c} = c^{\log_b a}$

$\log_x 0.008 = -\dfrac{3}{2}$

$0.008 = \dfrac{1}{125} = 5^{-3}$

$x^{-3/2} = 5^{-3}$, so $x^{3/2} = 125$

$x = 125^{2/3} = 25$

Trap: Dropping the minus sign: $x = 0.04$.

2 Peel nested logs

Work from the outside. The base jumps across as a power.

$\log_2(\log_3(\log_x 729)) = 0$

$\log_3(\log_x 729) = 2^0 = 1$

$\log_x 729 = 3^1 = 3$

$x^3 = 729$, **so** $x = 9$

Trap: Log of 0 instead of log of 1: 729.

3 Digits by logs

Digits of N = integer part of $\log N$, plus 1.

$\log 2 = 0.301$, $\log 3 = 0.477$

$\log 6^{20} = 20 \times 0.778 = 15.56$

$15 + 1 = 16$ **digits**

Trap: Forgetting the $+1$: 15.

4 Powers in the base and the argument

Argument's power on top, base's power below.

$\log_8 25 \times \log_5 32$

$= \dfrac{2}{3} \log_2 5 \times 5 \log_5 2$

$\log_2 5 \times \log_5 2 = 1$

$= \dfrac{10}{3}$

Trap: Multiplying by the base's 3: 30.

5 The chain

$\log_a b \times \log_b c = \log_a c$: the middle base cancels.

$\log_5 x \times \log_x 7 \times \log_7 y = 3$

$\log_5 y = 3$

$y = 5^3 = 125$

Trap: $3^5 = 243$.

Log equations

1 $t = \log$, then a quadratic

Product of the x values = base to the sum of the t values.

$\log_2 x + 6\log_x 2 = 5$

$t + \dfrac{6}{t} = 5$

$t^2 - 5t + 6 = 0$: $t = 2$ or $t = 3$

Product of $x = 2^{2+3} = 32$

Trap: Multiplying the t values: 6.

2 Reciprocals add

Flip each $\dfrac{1}{\log_k N}$ into $\log_N k$, then multiply the ks.

$\dfrac{1}{\log_4 N} + \dfrac{1}{\log_{16} N} = \dfrac{3}{2}$

$\log_N 4 + \log_N 16 = \log_N 64$

$N^{3/2} = 64$

$N = 64^{2/3} = 16$

Trap: Not flipping: 4.

3 A log in the power

Make the outer base a power of the log's base.

$4^{\log_2 3} = (2^{\log_2 3})^2 = 3^2 = 9$

$9^{\log_3 2} = (3^{\log_3 2})^2 = 2^2 = 4$

$4^{\log_2 3} + 9^{\log_3 2} = 13$

Trap: $4^{\log_2 3} = 3$: 5.

4 The swap

$x^{\log_2 5} = 5^{\log_2 x}$: the two terms are equal.

$5^{\log_2 x} + x^{\log_2 5} = 50$

$2 \times 5^{\log_2 x} = 50$

$5^{\log_2 x} = 25$, so $\log_2 x = 2$

$x = 4$

Trap: Stopping at $\log_2 x = 2$: 2.

5 The base must be allowed

Base > 0 and $\neq 1$; argument > 0. Test every root.

$\log_{x+1}(5x + 11) = 2$

$(x+1)^2 = 5x + 11$

$x^2 - 3x - 10 = 0$: $x = 5$ or $x = -2$

$x = -2$ makes the base -1: sum $= 5$

Trap: Keeping $x = -2$: 3.

6 A base below 1 flips the sign

For a base between 0 and 1, a bigger argument gives a smaller log.

$\log_{1/2}(n - 3) > -3$

$0 < n - 3 < (\frac{1}{2})^{-3} = 8$

$n = 4$ to 10: 7 integers

Trap: Forgetting $n - 3 > 0$: 10.

7 Before you answer

Primes first, one prime at a time · a^x is never 0 or negative · Turn every t back into x · A log base is positive and not 1 · A base below 1 flips an inequality.