Two equations

1 Look before you solve

The question often asks for a sum, a difference or a mix of the equations. Get that first; solve for x and y only if you must.

Same coefficient, same sign	Subtract the equations
Same coefficient, opposite sign	Add the equations
Coefficients swapped	Add for $x+y$, subtract for $x-y$

$13r + 7d = 1410$ and $7r + 13d = 1590$ (rice and dal, per kg)

Add: $20(r+d) = 3000$, so $r + d = 150$

Subtract: $6(d-r) = 180$, so $d - r = 30$

Rice $r = \dfrac{150-30}{2} = ₹60$ **a kg**

Trap: Mixing up which is bigger: ₹90 is the dal.

2 Coefficients that step evenly

Every coefficient up by the same gap? Subtract, then divide by the gap.

$3t + 5s + 7v = 245$

$5t + 7s + 9v = 335$

Subtract: $2t + 2s + 2v = 90$

$t + s + v = ₹45$

Trap: Stopping at 90.

3 Build the asked combination

Three unknowns, two equations: find λ, μ so that $\lambda E_1 + \mu E_2$ is what is asked.

$2x + 3y - z = 13,\ x - y + 2z = 4$

$2E_1 + E_2$: $5x + 5y + 0z = 30$

$x + y = 6$

Trap: "Cannot be determined".

4 One, none or infinitely many

One	$\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$: lines cross
Infinite	$\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$: one line
None	$\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$: parallel

$3x + 2y = 7,\ 12x + (a+3)y = b$: infinitely many

$\dfrac{12}{3} = \dfrac{a+3}{2} = \dfrac{b}{7} = 4$

$a + 3 = 8$ and $b = 28$

$a + b = 5 + 28 = 33$

Trap: Leaving the constant at 7: 12.

5 No solution: reject the merge

Solve $a_1 b_2 = a_2 b_1$, then test each root on the constants.

$(a-1)x + 4y = 8,\ 3x + (a+3)y = 12$

$(a-1)(a+3) = 12$, so $a = 3$ or $a = -5$

$a = 3$: $\dfrac{2}{3} = \dfrac{4}{6} = \dfrac{8}{12}$, one line

$a = -5$: $\dfrac{-6}{3} = \dfrac{4}{-2} \neq \dfrac{8}{12}$, **no solution**

Trap: Giving both values.

Whole numbers and squares

1 Count the whole-number solutions

For $ax + by = c$, find one solution, then step: x moves by $\frac{b}{g}$ and y by $\frac{a}{g}$, where $g = \gcd(a, b)$.

Pens ₹5, notebooks ₹8: $5p + 8n = 200$, at least one of each

$8n$ must be a multiple of 5, so $n = 5, 10, 15, 20$

Then $p = 32, 24, 16, 8$

4 ways

Trap: Allowing $n = 0$ or $p = 0$: 6.

2 Add to both sides, then factor

$xy + px + qy = r$ becomes $(x + q)(y + p) = r + pq$.

$xy + 2x + 3y = 29$

$(x + 3)(y + 2) = 29 + 6 = 35$

5×7 or 7×5: $(x, y) = (2, 5)$ or $(4, 3)$

$x + y = 7$ both ways

Trap: $5 + 7 = 12$ without shifting back.

3 A sum of squares equal to 0

Squares are never negative, so each one must be 0.

$a^2 + b^2 + c^2 + 29 = 4a - 6b + 8c$

$(a - 2)^2 + (b + 3)^2 + (c - 4)^2 = 0$

$a = 2,\ b = -3,\ c = 4$

$a + b + c = 3$

Trap: A sign slip, $b = 3$: 9.

4 The least value of a sum of squares

Complete each square; what is left over is the floor.

$x^2 - 4x = (x - 2)^2 - 4$

$y^2 + 6y = (y + 3)^2 - 9$

$x^2 + y^2 - 4x + 6y + 20 = (x - 2)^2 + (y + 3)^2 + 7$

Least value 7, at $x = 2,\ y = -3$

Trap: Adding the 4 and 9: 33.

5 A quadratic hiding in $|x|$

Put $t = |x| \geq 0$. Each positive t gives $x = \pm t$.

$x^2 - 5|x| + 6 = 0$

$t^2 - 5t + 6 = 0$, so $t = 2$ or 3

$x = \pm 2,\ \pm 3$

Product of the solutions $= 36$

Trap: Stopping at t: $2 \times 3 = 6$.

6 Quick checks

Ask for the combination, not the unknowns.

Equal ratios? Check the constants before you say "infinitely many".

Positive integers start at 1, non-negative ones at 0; check every case against the condition and the word "distinct".

Roots of a quadratic

1 The discriminant decides

$$x = \frac{-b \pm \sqrt{D}}{2a}, \qquad D = b^2 - 4ac$$

$D > 0$	Two real roots; rational when D is a perfect square and a, b, c are rational
$D = 0$	One repeated root, $x = -\frac{b}{2a}$
$D < 0$	No real root

Split $6x^2 - 7x - 20$: two numbers with product $ac = -120$ and sum -7

They are -15 and 8, so the roots are $\frac{15}{6}$ and $-\frac{8}{6}$

Positive root $= \frac{5}{2}$

Trap: Not dividing by $a = 6$: 15.

2 No real root: $D < 0$

$x^2 - kx + 9 = 0$

$D = k^2 - 36 < 0$

$-6 < k < 6$

11 **integers**

Trap: Keeping $k = \pm 6$: 13.

3 Rational roots need a square D

$2x^2 + 7x + c = 0$: $D = 49 - 8c$

$c = 1$ to 6: $41, 33, 25, 17, 9, 1$

Squares at $c = 3, 5, 6$

Sum of all $c = 14$

Trap: $D \geq 0$ only: 21.

4 Irrational roots come in pairs

Rational coefficients: $p - \sqrt{q}$ brings $p + \sqrt{q}$.

Root $3 - \sqrt{5}$ of $x^2 + bx + c = 0$

Sum 6, so $b = -6$

Product $9 - 5 = 4$, so $c = 4$

$b + c = -2$

Trap: Product as $9 + 5$: 8.

5 Roots in a ratio

Write the roots as k and $2k$; the sum fixes k.

$x^2 - 18x + c = 0$, one root twice the other

$3k = 18$, so $k = 6$

$c = 6 \times 12 = 72$

Trap: 9 as the smaller root: 162.

6 Common root: subtract to kill x^2

The difference of the two equations is linear, so it gives the common root at once. Then put that root back into either equation.

$x^2 - ax + 6 = 0$ and $x^2 - (a+1)x + 9 = 0$

Subtract: $x - 3 = 0$, so the common root is 3

$9 - 3a + 6 = 0$

$a = 5$

Trap: A sign slip gives root -3 and $a = -5$.

Sum, product and the graph

1 Sum and product, not the roots

For $ax^2 + bx + c = 0$: $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$, $\alpha - \beta = \frac{\sqrt{D}}{a}$ (larger minus smaller, $a > 0$).

$\alpha^2 + \beta^2$	$(\alpha + \beta)^2 - 2\alpha\beta$
$\alpha^3 + \beta^3$	$(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$2x^2 - 8x + 3 = 0$: $\alpha + \beta = 4$, $\alpha\beta = \frac{3}{2}$

$\alpha^2 + \beta^2 = 16 - 3$

$= 13$

Trap: Dropping the 2: $16 - \frac{3}{2} = 14.5$.

2 Sum of cubes

$x^2 - 6x + 4 = 0$: $s = 6, p = 4$

$s^3 - 3ps = 216 - 72$

$\alpha^3 + \beta^3 = 144$

Trap: $216 - 12 = 204$.

3 New roots, new equation

$x^2 - 4x + 1 = 0$: $s = 4, p = 1$

Squared roots: sum $s^2 - 2p = 14$, product $p^2 = 1$

$x^2 - 14x + 1 = 0$

Trap: $s^2 = 16$ as the new sum.

4 The graph of $ax^2 + bx + c$

$a > 0$ opens up (a least value), $a < 0$ opens down (a greatest value). The vertex is at $x = -\frac{b}{2a}$ with value $-\frac{D}{4a}$. The curve cuts the x-axis at the roots and the y-axis at c.

Daily profit $-2x^2 + 48x - 160$ at a price of ₹x

Vertex $x = \frac{48}{4} = 12$

$-288 + 576 - 160$

Highest profit ₹128

Trap: $x = \frac{48}{2} = 24$: a loss of ₹160.

5 Symmetry finds the vertex

Equal values sit equally far from the axis.

$f(x) = x^2 + bx + c$, $f(2) = f(8)$, $f(0) = 7$

Axis $x = \frac{2 + 8}{2} = 5$, so $b = -10$; $c = 7$

$f(5) = 25 - 50 + 7$

Least value -18

Trap: Axis at 3: -2.

6 Negative between the roots

For $a > 0$ the curve dips below the axis only between the roots.

$x^2 + bx + c < 0$ exactly for $-2 < x < 5$

$(x + 2)(x - 5) = x^2 - 3x - 10$

$b + c = -3 - 10 = -13$

Trap: $b = +3$: -7.